Exam 4

Name: ______________________________ Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1	A	B	C	D	E	**6**	A	B	C	D	E
2	A	B	C	D	E	**7**	A	B	C	D	E
3	A	B	C	D	E	**8**	A	B	C	D	E
4	A	B	C	D	E	**9**	A	B	C	D	E
5	A	B	C	D	E	**10**	A	B	C	D	E

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Trig identities

- $\sin^2(x) + \cos^2(x) = 1$,
- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Multiple Choice Questions

1. (5 points) Use the midpoint rule with 4 intervals to approximate $\int_0^2 (4 - x^2)dx$.

   A. $\frac{1}{2}(\frac{16}{4} + \frac{15}{4} + \frac{12}{4} + \frac{7}{4})$.
   B. $\frac{1}{2}(\frac{63}{16} + \frac{55}{16} + \frac{39}{16} + \frac{15}{16})$
   C. $\frac{1}{4}(\frac{16}{4} + \frac{30}{4} + \frac{24}{4} + \frac{14}{4})$.
   D. $\frac{1}{6}(\frac{16}{4} + \frac{60}{4} + \frac{24}{4} + \frac{28}{4})$.
   E. 0.

2. (5 points) The improper integral $\int_2^\infty \frac{1}{x^p \ln x} dx$ ...

   A. converges for $p = 2$ and diverges for $p = 0, 1$.
   B. converges for $p = 2, 1$ and diverges for $p = 0$.
   C. converges for $p = 2, 1, 0$.
   D. diverges for $p = 0, 1, 2$.
   E. converges for $p = 1$ and diverges for $p = 0, 2$.

Solution: If $u = \ln x$ and $du = \frac{1}{x}dx$,

$$\int \frac{1}{x}\frac{1}{\ln x}dx = \int \frac{1}{u}du = \ln(u) + C = \ln(\ln(x)) + C$$

so this integral diverges. If $p = 0$ the integral diverges by comparison with $\frac{1}{x \ln x}$. If $p = 2$ the integral converges by comparison with $\frac{1}{x^2}$.

3. (5 points) Which of the following sequences converge?

   A. $b_n = 5^n$.

   B. $c_n = \dfrac{(-1)^n n}{n+1}$.

   C. $a_n = \dfrac{5n-1}{12n+9}$.

   D. None of the above.

   E. All of the above.

4. (5 points) Find the Taylor polynomial $T_3(x)$ for e^{3x} centered at 0. What is $T_3(1)$?

   A. $1 + 1 + \dfrac{1}{2} + \dfrac{1}{6}$.

   B. $1 + 3 + 9 + 27$.

   C. $1 + 3 + \dfrac{9}{2} + \dfrac{27}{3}$.

   D. $1 + 3 + \dfrac{9}{2} + \dfrac{27}{6}$.

   E. 1.

Solution:

$$\begin{aligned} f(x) &= e^{3x} & f(0) &= 1 \\ f'(x) &= 3e^{3x} & f'(0) &= 3 \\ f''(x) &= 3^2 e^{3x} & f''(0) &= 9 \\ f'''(x) &= 3^3 e^{3x} & f'''(0) &= 27 \end{aligned}$$

5. (5 points) Let $f(x) = \sqrt{x}$. Find a value of $c \in [4,9]$ so that $f(c)$ is the average value of $f(x)$ on the interval $[4,9]$.

   A. $\left(\dfrac{38}{15}\right)^2$.

   B. $\dfrac{38}{15}$.

   C. $\left(\dfrac{27}{15}\right)^2$.

   D. $\dfrac{-8}{15}$.

   E. $\left(\dfrac{38}{5}\right)^2$.

Solution: The average value is

$$\frac{1}{5}\int_4^9 x^{\frac{1}{2}}\,dx = \frac{1}{5}\;\frac{2}{3}x^{\frac{3}{2}}\Big|_4^9 = \frac{1}{5}\frac{2}{3}(9^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{1}{5}\frac{2}{3}(27-8) = \frac{38}{15}$$

6. (5 points) Find the volume of a solid whose base is the unit circle $x^2 + y^2 = 1$ and the cross sections perpendicular to the x-axis are triangles whose height and base are equal.

A. $\frac{4}{3}$.

B. $\frac{8}{3}$.

C. 0

D. $\frac{2}{3}$.

E. $\frac{16}{3}$.

Solution: The volume is

$$\int_{-1}^{1} \frac{1}{2}(2\sqrt{1-x^2})^2 dx = 2\int_{-1}^{1}(1-x^2)dx = 2\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1}$$
$$= 2\left(\left(1-\frac{1}{3}\right) - \left(-1 - \frac{-1}{3}\right)\right) = 2\left(\frac{2}{3}+\frac{2}{3}\right)$$

7. (5 points) A surface is created by rotating the graph of $f(x) = 5+\sin(6x)$ from $x = 0$ to $x = 100$ around the x-axis. What is the integral that computes the area of this surface?

A. $\int_0^{100} 2\pi(5+\sin(6x))dx$.

B. $\int_0^{100} 2\pi\sqrt{1+(6\cos(6x))^2}dx$.

C. $\int_0^{100} 2\pi(5+\sin(6x))\sqrt{1+6\cos(6x)}dx$.

D. $\int_0^{100} 2\pi(5+6\cos(6x))\sqrt{1+\sin(6x)}dx$.

E. $\int_0^{100} 2\pi(5+\sin(6x))\sqrt{1+(6\cos(6x))^2}dx$.

Solution: $f'(x) = 6\cos(6x)$

8. (5 points) Find the center of mass of the region between the curves $y = x$ and $y = x^2$ (Assume the region has constant density).

A. $\left(\frac{1}{2}, \frac{2}{5}\right)$**.**

B. $\left(\frac{1}{12}, \frac{1}{15}\right)$.

C. $\left(\frac{2}{5}, \frac{1}{2}\right)$.

D. $\left(\frac{1}{15}, \frac{1}{12}\right)$.

E. $\left(1, \frac{1}{5}\right)$.

Solution:

$$A = \int_0^1 (x - x^2)dx = \frac{x^2}{2} - \frac{x^3}{3}\bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

$$\overline{x} = 6\int_0^1 x(x - x^2)dx = 6\int_0^1 x^2 - x^3 dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\bigg|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = 6\frac{4-3}{12} = \frac{1}{2}$$

$$\overline{y} = 6\int_0^1 \frac{1}{2}(x^2 - x^4)dx = 3\left(\frac{x^3}{3} - \frac{x^5}{5}\right)\bigg|_0^1 = 3\left(\frac{1}{3} - \frac{1}{5}\right) = 3\left(\frac{5-3}{15}\right) = \frac{2}{5}$$

9. (5 points) Which of the following is the equation for a hyperbola with vertices $(0, \pm 2)$ and foci $(0, \pm 5)$?

   A. $\dfrac{x^2}{4} - \dfrac{y^2}{21} = 1.$

   B. $\dfrac{y^2}{4} + \dfrac{x^2}{21} = 1.$

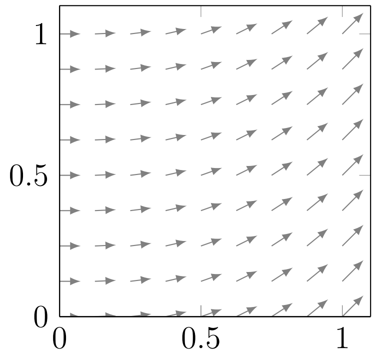
   C. $\dfrac{y^2}{4} - \dfrac{x^2}{21} = 1.$

   D. $\dfrac{x^2}{4} + \dfrac{y^2}{21} = 1.$

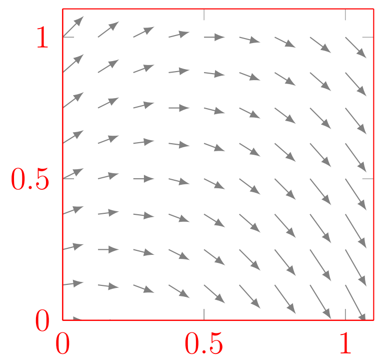
   E. $\dfrac{x^2}{21} - \dfrac{y^2}{4} = 1.$

Solution: $b^2 = c^2 - a^2 = 5^2 - 2^2 = 25 - 4 = 21$

10. (5 points) Which of the following is the direction field for the equation $y' = y - 2x$?



A.



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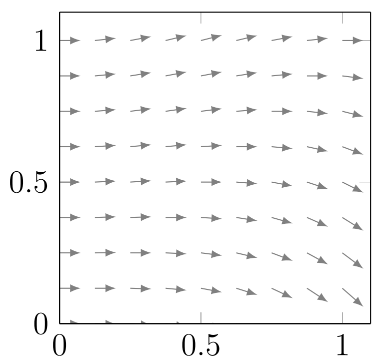

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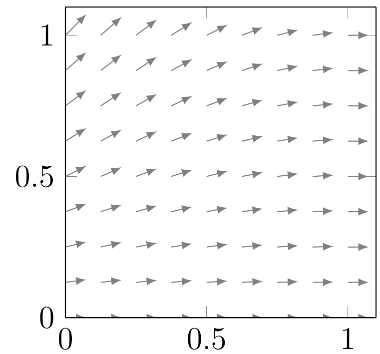

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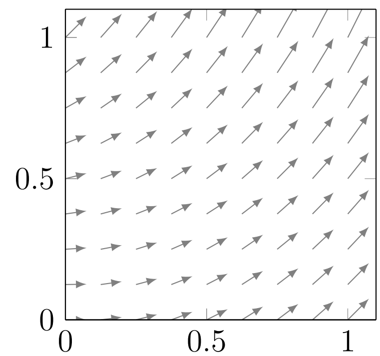


E.

Free Response Questions

11. Compute the following integrals.

(a) (5 points) $\int x \sin x \cos x dx$

Solution: Let $u = x$ and $dv = \sin x \cos x dx$. Then $du = dx$ and $v = \frac{1}{2}\sin^2 x$

$$\begin{aligned}\int x \sin x \cos x dx &= \frac{1}{2} x \sin^2 x - \int \frac{1}{2}\sin^2 x dx \\ &= \frac{1}{2} x \sin^2 x - \frac{1}{4}\int (1 - \cos(2x)) dx \\ &= \frac{1}{2} x \sin^2 x - \frac{1}{4}(x - \frac{1}{2}\sin(2x)) + C\end{aligned}$$

(b) (5 points) $\int \frac{dx}{x^2\sqrt{x^2-4}}$

Solution: Take $x = \sqrt{2}\sec\theta$. Then $dx = \sqrt{2}\sec\theta\tan\theta d\theta$.

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-2}} &= \int \frac{\sqrt{2}\sec\theta\tan\theta d\theta}{(\sqrt{2}\sec\theta)^2\sqrt{(\sqrt{2}\sec\theta)^2 - 2}} \\ &= \int \frac{\sqrt{2}\sec\theta\tan\theta d\theta}{2\sec^2\theta\sqrt{2\sec^2\theta - 2}} = \int \frac{\tan\theta d\theta}{2\sec\theta\tan\theta} = \frac{1}{2}\int \cos\theta d\theta \\ &= \frac{1}{2}\sin\theta + C = \frac{1}{2}\frac{\sqrt{x^2-2}}{x} + C\end{aligned}$$

12. Determine if the following series converge or diverge. Justify your answer!

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^{\frac{3}{2}}}$.

Solution: Compute

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{1+n^{\frac{3}{2}}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+n^{\frac{3}{2}}} n = \lim_{n\to\infty} \frac{n^{\frac{3}{2}}}{1+n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{\frac{3}{2}n^{\frac{1}{2}}}{\frac{3}{2}n^{\frac{1}{2}}} = 1.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges this series diverges by the limit comparison test.

(b) (5 points) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{k}{3k+1}\right)^{k}$.

Solution:

$$\lim_{k\to\infty} \left(\left(\frac{k}{3k+1}\right)^{k}\right)^{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{3k+1} = \frac{1}{3}$$

So the series converges.

13. Let S be the solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ around the line $x = 3$.

(a) (3 points) Set up the integral that computes the volume of S using the disk/washer method.

Solution:

$$\int_0^1 \pi((3 - y^2)^2 - (3 - \sqrt{y})^2)dy$$

(b) (3 points) Set up the integral that computes the volume of S using the cylindrical shells method.

Solution:

$$\int_0^1 2\pi(3 - x)(\sqrt{x} - x^2)dx$$

(c) (4 points) Choose one of these integrals and find the volume of S.

Solution:

$$\begin{aligned}\int_0^1 \pi((3 - y^2)^2 - (3 - \sqrt{y})^2)dy &= \int_0^1 \pi(9 - 6y^2 + y^4 - 9 + 6\sqrt{y} - y)dy \\ &= \int_0^1 \pi(-6y^2 + y^4 + 6\sqrt{y} - y)dy \\ &= \pi(-2y^3 + \frac{1}{5}y^5 + 4y^{\frac{3}{2}} - \frac{1}{2}y^2)dy = \pi(-2 + \frac{1}{5} + 4 - \frac{1}{2}) \\ &= \pi\frac{20 + 2 - 5}{10} = \pi\frac{17}{10}\end{aligned}$$

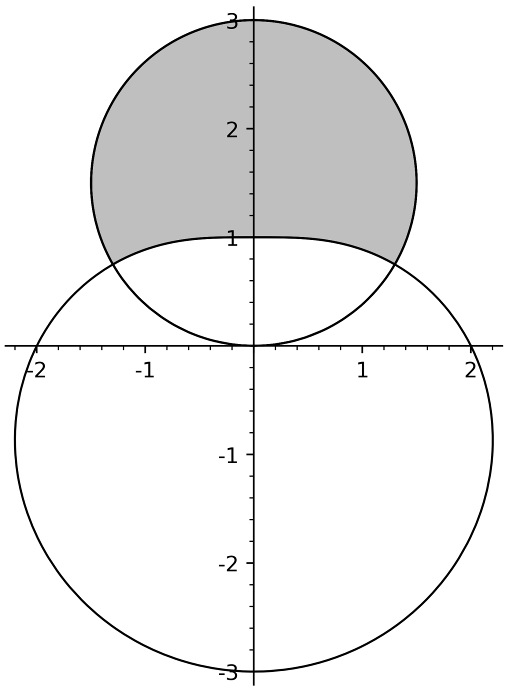
$$\begin{aligned}\int_0^1 2\pi(3 - x)(\sqrt{x} - x^2)dx &= \int_0^1 2\pi(3\sqrt{x} - 3x^2 - x^{\frac{3}{2}} + x^3)dx \\ &= 2\pi\left(2x^{\frac{3}{2}} - x^3 - \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{4}x^4\right)|_0^1 \\ &= 2\pi(2 - 1 - \frac{2}{5} + \frac{1}{4}) = 2\pi\frac{20 - 8 + 5}{20} = \pi\frac{17}{10}\end{aligned}$$

14. (a) (5 points) Find the slope of the tangent to the parametric curve given by $x = t\sin t$, $y = t\cos t$ at the point $(0, -\pi)$.

Solution: $\frac{dx}{dt} = \sin t + t\cos t$ and $\frac{dy}{dt} = \cos t - t\sin t$ so

$$\frac{dy}{dx} = \frac{\cos t - t\sin t}{\sin t + t\cos t}$$

(b) (5 points) Write an integral that computes the area of the region (shown below) that is inside the polar curve $r = 3\sin\theta$ and outside the polar curve $r = 2 - \sin\theta$.



Solution:

$$3\sin\theta = 2 - \sin\theta$$
$$4\sin\theta = 2$$
$$\sin\theta = \frac{1}{2}$$

then $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}((3\sin\theta)^2 - (2 - \sin\theta)^2)d\theta$$

15. (a) (5 points) For what values of k does $y = e^{kt}$ satisfy the differential equation

$$y'' - y' - 2y = 0?$$

Solution: $y = ke^{kt}$ and $y'' = k^2e^{kt}$ and so we need

$$0 = k^2e^{kt} - ke^{kt} - 2e^{kt} = e^{kt}(k^2 - k - 2) = e^{kt}(k+1)(k-2)$$

This holds if $k = -1$ or $k = 2$.

(b) (5 points) Find the solution to the differential equation $y' = y^2x$ that satisfies the initial condition $y(0) = 1$.

Solution: Let $f(y) = y^2$ and $g(x) = x$ then

$$\int \frac{1}{y^2}dy = \int y^{-2}dy = -y^{-1} + C_1$$

$$\int xdx = \frac{1}{2}x^2 + C_2$$

$$\frac{-1}{y} = \frac{x^2 + C}{2}$$

$$y = \frac{-2}{x^2 + C}$$

Then

$$1 = \frac{-2}{0^2 + C} = \frac{-2}{C}$$

So $C = -2$ and $y = \frac{-2}{x^2-2}$.