Exam 4

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Find $\int x^3 \ln(x)\, dx$.

   A. $\frac{1}{4}x^3 + C$

   B. $\frac{1}{4}x^4 \ln(x) - \frac{1}{5}x^5 \ln(x) + C$

   C. $\frac{1}{8}x^4(\ln(x))^2 + C$

   D. $\frac{1}{4}x^4 \ln(x) - \frac{1}{16}x^4 + C$

   E. $x^3 \ln(x) - \frac{3}{2}x^2 + C$

2. (5 points) Find $\int \sin^5(x)\, dx$.

   A. $-\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + C$

   B. $-\cos(x) - \frac{1}{5}\cos^5(x) + C$

   C. $\frac{1}{6}\sin^6(x)\cos(x) + C$

   D. $-1 + 2\cos^2(x) - \cos^4(x) + C$

   E. $-\cos(x) + \cos^2(x) - \frac{1}{3}\cos^3(x) + C$

3. (5 points) Which of the following is equal to the integral

$$\int \frac{5dx}{x^2\sqrt{x^2+100}}$$

after making the substitution $x = 10\tan(\theta)$?

A. $\int \frac{\cos^3(\theta)d\theta}{200\sin^2(\theta)}$

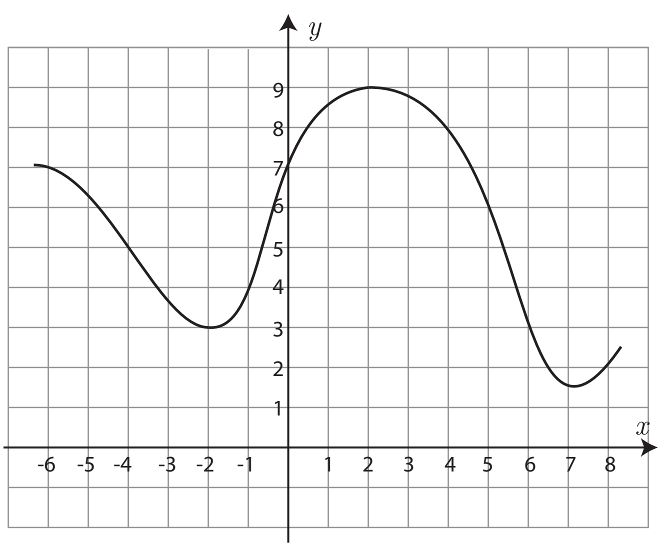
B. $\int \frac{d\theta}{200\tan^3(\theta)+200}$

C. $\int \frac{\cos(\theta)d\theta}{20\sin^2(\theta)}$

D. $\int \frac{\sec^2(\theta)d\theta}{20\tan^2(\theta)(\tan(\theta)+1)}$

E. $\int \frac{1}{20}\csc^2(\theta)d\theta$

4. (5 points) Apply the midpoint rule to estimate the integral $\int_{-4}^{8} f(x)\, dx$ using **three** intervals (i.e. find M_3), where the graph of $f(x)$ is given below.



A. 60

B. 66

C. 67.5

D. 68

E. 80

5. (5 points) Find the sum of the geometric series

$$2 + \frac{2}{5} + \frac{2}{25} + \frac{2}{125} + \frac{2}{625} \cdots$$

A. $\frac{10}{3}$
B. $\frac{5}{2}$
C. $\frac{8}{5}$
D. $\frac{1512}{625}$
E. 0

6. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{5(x-3)^n}{n}$?

A. $\{3\}$
B. $(-2, 8)$
C. $[2, 4)$
D. $[\frac{14}{5}, \frac{16}{5}]$
E. $(-\infty, \infty)$

7. (5 points) The curve $y = \ln(x)$ from $x = 1$ to $x = 9$ is rotated about the $\boldsymbol{y}$**−axis**. Which integral computes the **surface area** of the resulting surface?

A. $\displaystyle\int_1^9 2\pi \ln(x)\sqrt{1+\frac{1}{x^2}}dx$

B. $\displaystyle\int_1^9 2\pi x\sqrt{1+(\ln(x))^2}dx$

C. $\displaystyle\int_1^9 2\pi\left(1+\frac{1}{x}\right)dx$

D. $\displaystyle\int_1^9 2\pi x\sqrt{1+\frac{1}{x^2}}dx$

E. $\displaystyle\int_1^9 2\pi\frac{\ln(x)}{1+x}dx$

8. (5 points) Consider the curve C parametrized by $x(t) = 5t+1$ and $y(t) = t^3+7t^2$. Find the slope of the tangent line to C at the point $(x, y) = (-4, 6)$.

A. $-\frac{11}{5}$

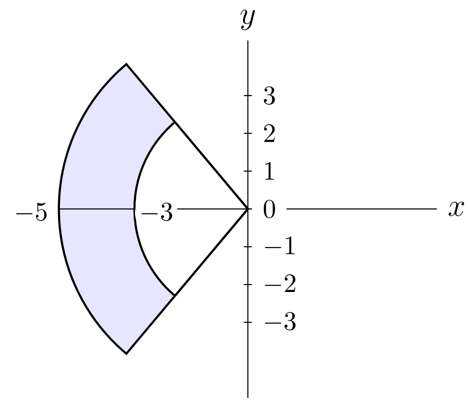
B. $-\frac{3}{2}$

C. -1

D. $\frac{17}{5}$

E. $\frac{192}{5}$

9. (5 points) Which description in polar coordinates matches the shaded area in the following graph?



A. $-5 \leq r \leq -3,\ 0 \leq \theta \leq \frac{\pi}{2}$

B. $r \geq 0,\ \frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{3}$

C. $r \geq 3,\ \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

D. $3 \leq r \leq 5,\ \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

E. $r \leq 5,\ -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

10. (5 points) Find the **vertices** of the hyperbola with equation $\dfrac{(x-3)^2}{36} - \dfrac{(y-2)^2}{49} = 1$.

A. $(6, 0)$ and $(-6, 0)$

B. $(3, 9)$ and $(3, -5)$

C. $(9, 2)$ **and** $(-3, 2)$

D. $(3+\sqrt{85}, 2)$ and $(3-\sqrt{85}, 2)$

E. $(3, 7+\sqrt{85})$ and $(3, 7-\sqrt{85})$

Free Response Questions

11. (a) (4 points) Evaluate $\int \frac{2x-7}{x^2+25}dx$.

Solution:

$$\int \frac{2x-7}{x^2+25}dx = \int \frac{2x}{x^2+25}dx + \int \frac{-7}{x^2+25}dx = \ln(x^2+25) - \frac{7}{5}\arctan\left(\frac{x}{5}\right) + C$$

(b) (6 points) Find a partial fraction decomposition for the function below. (Note: You DO need to solve for the coefficients, but you do NOT need to compute an integral.)

$$\frac{7x^2+x+20}{x(x^2+5)}$$

Solution:

$$\frac{7x^2+x+20}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5}$$

Solving for the coefficients, we have

$$7x^2+x+20+A(x^2+5)+(Bx+C)x.$$

Equating coefficients of $x^2, x, 1$ on both sides yields three equations $7 = A+B$, $1 = C$, $20 = 5A$. We conclude that $A = 4, B = 3, C = 1$, so the final answer is

$$\frac{4}{x} + \frac{3x+1}{x^2+5}.$$

12. (a) (5 points) Write the Maclaurin series (i.e., the Taylor series centered at $x = 0$) for the function

$$f(x) = x^3 \sin(3x^2).$$

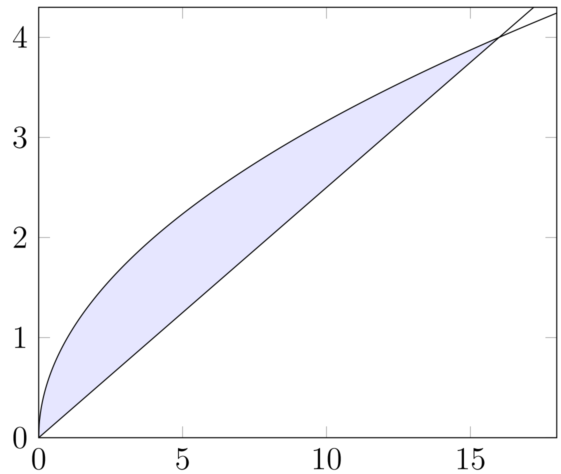
Solution:

$$\begin{aligned} x^3 \sin(3x^2) &= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n (3x^2)^{2n+1}}{(2n+1)!} \\ &= x^3 \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{4n+2}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} x^{4n+5}}{(2n+1)!} \end{aligned}$$

(b) (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^5+1}$ converge or diverge? Show all steps to justify your answer, and clearly list which test(s) you use.

Solution:
Observe that $\frac{\sin^2(n)}{n^5+1} \leq \frac{1}{n^5+1} < \frac{1}{n^5}$. Since the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges by p-series with $p = 5 > 1$, by the comparison test we conclude that $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^5+1}$ also converges.

13. The region between $y = \sqrt{x}$ and $y = \frac{x}{4}$ is shown below. Let V be the solid obtained by rotating this region about the $\boldsymbol{x}$**−axis**.



(a) (5 points) Set up but do not evaluate the integral that computes the volume of V using the **disk/washer** method.

Solution:

$$V = \pi \int_0^{16} \left((\sqrt{x})^2 - \left(\frac{x}{4}\right)^2 \right) \, dx$$

(b) (5 points) Set up but do not evaluate the integral that computes the volume of V using the **shell** method.

Solution:

$$V = 2\pi \int_0^4 y(4y - y^2) \, dy$$

14. (a) (6 points) Find the area in the first quadrant (i.e., for $0 \leq \theta \leq \frac{\pi}{2}$) within the polar curve $r = 2 + \cos(\theta)$. Clearly set up an integral, and **show all steps to evaluate the integral**.

Solution:

$$\begin{aligned} A &= \int_a^b \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2}(2+\cos\theta)^2 \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{2}(4 + 4\cos\theta + \cos^2\theta) \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{2}(4 + 4\cos\theta + \frac{1}{2} + \frac{1}{2}\cos(2\theta)) \, d\theta \\ &= \frac{1}{2}(4\theta + 4\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta))\Big|_0^{\pi/2} \\ &= \frac{9\pi}{8} + 2. \end{aligned}$$

(b) (4 points) Find a Cartesian equation satisfied by the curve parametrized by $x(t) = t^2$ and $y(t) = \frac{1}{4t}$.

Solution: Since $y(t) = \frac{1}{4t}$ then $t = \frac{1}{4y}$, so $x = \left(\frac{1}{4y}\right)^2$. Equivalently, since $x = t^2$, then $t = \sqrt{x}$, so $y = \frac{1}{4\sqrt{x}}$ with $x \geq 0$.

15. (a) (4 points) Write the equation of the parabola which has vertex $(2, 5)$ and focus $(2, 1)$.

Solution: $y = \frac{1}{4p}(x-h)^2 + k = \frac{1}{4(-4)}(x-2)^2 + 5$.

(b) (6 points) Find the foci of the ellipse defined by the equation

$$9x^2 + y^2 - 10y + 16 = 0.$$

Solution:

$$\begin{aligned} 9x^2 + y^2 - 10y + 25 - 25 + 16 &= 0 \\ 9x^2 + (y-5)^2 &= 9 \\ \frac{x^2}{1} + \frac{(y-5)^2}{9} &= 1 \end{aligned}$$

Thus, the center is $(0, 5)$, and $c^2 = 9 - 1 = 8$, so $c = \sqrt{8}$. Then the foci are at $(0, 5+\sqrt{8})$ and $(0, 5-\sqrt{8})$.