MA 114 — Calculus II Spring 2015
Exam 4 May. 4, 2015

Name: KEY

Section: ____________

Last 4 digits of student ID #: ________

- No books or notes may be used.
- Turn off all your electronic devices and do not wear ear-plugs during the exam.
- You may use a calculator, but not one which has symbolic manipulation capabilities or a QWERTY keyboard.
- Additional blank sheets for scratch work are available upon request.
- **Multiple Choice Questions:**
  Record your answers on the right of this cover page by marking the box corresponding to the correct answer.
- **Free Response Questions:**
  Show all your work on the page of the problem. Clearly indicate your answer and the reasoning used to arrive at that answer.

Multiple Choice Answers

Question					
1	A		C	D	E
2	A	B		D	E
3		B	C	D	E
4	A	B	C		E
5	A	B	C	D	
6	A	B		D	E
7	A	B	C	D	

Exam Scores

Question	Score	Total
MC		28
8		15
9		15
10		10
11		12
12		20
Total		100

Unsupported answers for the free response questions may not receive credit!

Feel free to use the following identities:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}, \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}, \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

Record the correct answer to the following problems on the front page of this exam.

1. Find the antiderivative of $(x+1)\sin(2x)$:

A. $\cos(2x) + \frac{(x+1)^2}{2}\sin(2x) + C$

B. $\frac{-(x+1)}{2}\cos(2x) + \frac{\sin(2x)}{4} + C$

C. $(x+1)\sin(2x) - \cos(2x) + C$

D. $\sin(2x) - (x+1)\cos(2x) + C$

E. $\frac{x+1}{2}\cos(2x) - \frac{1}{4}\sin(2x) + C$

$u = x+1$ $\quad dv = \sin(2x)dx$

$du = dx$ $\quad v = \frac{-\cos(2x)}{2}$

$= \frac{-(x+1)}{2}\cos(2x) + \frac{1}{2}\int \cos(2x)dx$

$= \frac{-(x+1)}{2}\cos(2x) + \frac{\sin(2x)}{4} + C$

2. The rectangular coordinates for the point with polar coordinates $r = 3$ and $\theta = 2\pi/3$ are

A. $\left(3, \frac{2\pi}{3}\right)$

B. $\left(-\frac{2\pi}{3}, 3\right)$

C. $\left(\frac{-3}{2}, \frac{3\sqrt{3}}{2}\right)$

D. $\left(\sqrt{9 + \frac{4\pi^2}{9}}, \arctan\left(\frac{2\pi}{9}\right)\right)$

E. $(0, -3)$

$x = r\cos(\theta)$ $\quad y = r\sin(\theta)$

$= 3\cos\left(\frac{2\pi}{3}\right)$ $\quad = 3\sin\left(\frac{2\pi}{3}\right)$

$= 3 \cdot \left(-\frac{1}{2}\right)$ $\quad = 3 \cdot \frac{\sqrt{3}}{2}$

Record the correct answer to the following problems on the front page of this exam.

3. Which of the following improper integrals converge?

$$\text{I.} \int_1^\infty \frac{dx}{x^2}, \qquad \text{II.} \int_0^1 \frac{dx}{x^3}, \qquad \text{III.} \int_1^\infty \frac{dx}{(x-2)^4}$$

A. I. only

B. II. only

C. I. and III. only

D. all of these converge

E. none of these converge

I. converges by integral p-test

II. diverges by integral p-test

III $\int_1^\infty \frac{dx}{(x-2)^4} = \int_1^2 \frac{dx}{(x-2)^4} + \int_2^\infty \frac{dx}{(x-2)^4}$

both diverge

4. Which of the following is true for the infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$?

A. It is divergent.

B. It is absolutely convergent, but divergent.

C. It is convergent, but not absolutely convergent.

D. It is absolutely convergent and convergent.

E. None of the above.

Record the correct answer to the following problems on the front page of this exam.

5. The power series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}$$

is a Taylor series for which function?

$\ln(1+0) = \ln(1) = 0$

A. $\sin(-x)$

$\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1+x^2}$ $\quad f'(0) = 0$

B. $\arctan(-x)$

$\frac{d^2}{dx^2}\ln(1+x^2) = \frac{2(1+x^2) - 2x \cdot 2x}{1+x^2}$

C. $\ln(1-x)$

D. xe^{-x}

$= \frac{2-2x^2}{1+x^2}$ $\quad f''(0) = 2$

E. $\ln(1+x^2)$

So Taylor series for $\ln(1+x^2)$ starts

$f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots$

$= 0 + 0 + \frac{2}{2}x^2 + \cdots = x^2 + \cdots$

The other 4 functions have a nonzero linear coefficient.

6. What is the area of the surface obtained by rotating the curve $y = \frac{x^3}{3}$, for $0 \le x \le 1$, around the x-axis?

$f'(x) = \frac{3}{3}x^2 = x^2$

A. $\frac{\pi}{81}(10\sqrt{10}-1)$

$\text{Area} = 2\pi\int_0^1 \frac{x^3}{3}\sqrt{1+(x^2)^2}\,dx$

B. $\frac{4\pi}{45}(1+\sqrt{2})$

$= \frac{2\pi}{3}\int_0^1 x^3\sqrt{1+x^4}\,dx$

C. $\frac{\pi}{9}(2\sqrt{2}-1)$

$u = 1+x^4$
$du = 4x^3$

D. $\frac{110}{101}$

$= \frac{2\pi}{4\cdot 3}\int_1^2 \sqrt{u}\,du$

E. 6.8451

$= \frac{\pi}{6}\cdot\frac{2}{3}u^{3/2}\Big|_1^2 = \frac{\pi}{9}(2\sqrt{2}-1)$

Record the correct answer to the following problems on the front page of this exam.

7. Which of the following describes the behavior of the solution to the differential equation $y' = 3(y - 10)$ with initial condition $y(0) = 5$?

A. $\lim_{t\to\infty} y(t) = 3$

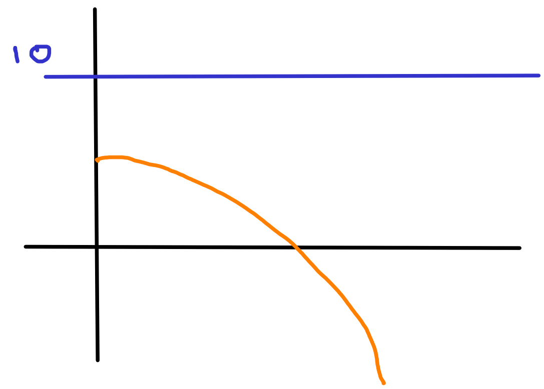
B. $\lim_{t\to\infty} y(t) = 5$

C. $\lim_{t\to\infty} y(t) = 10$

D. $\lim_{t\to\infty} y(t) = \infty$

E. $\lim_{t\to\infty} y(t) = -\infty$

$k = 3 > 0$ growth



8. Consider the curve parametrized by

$$x(t) = t^2 - 1, \qquad y(t) = t^3 - 3t.$$

(a) Find the value(s) of t that correspond to the point $(2, 0)$ under this parametrization.

5

1 $t^2 - 1 = 2$ and $t^3 - 3t = 0$ 1

$t^2 = 3$ $\quad t(t^2-3) = 0$

1 $t = \pm\sqrt{3}$ $\quad t = 0, \pm\sqrt{3}$ 1

Solutions are $t = \pm\sqrt{3}$ 1

(b) Find the equation for the tangent line(s) to the curve at the point $(2, 0)$.

6

slope $= \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2-3}{2t} = \frac{3}{2}\frac{t^2-1}{t}$ 1 1

@ $t = \sqrt{3}$ slope $= \frac{3}{2}\frac{(\sqrt{3})^2-1}{\sqrt{3}} = \frac{3}{2}\frac{3-1}{\sqrt{3}} = \sqrt{3}$ 1

equation is $y = \sqrt{3}(x-2)$ or $y = \sqrt{3}x - 2\sqrt{3}$ 1

@ $t = -\sqrt{3}$, slope $= \frac{3}{2}\frac{(-\sqrt{3})^2-1}{-\sqrt{3}} = \frac{3}{2}\frac{3-1}{-\sqrt{3}} = -\sqrt{3}$ 1

equation is $y = -\sqrt{3}(x-2)$ or $y = -\sqrt{3}x + 2\sqrt{3}$ 1

(c) Find the point(s) where the tangent line to the curve is vertical.

4

Slope $= \frac{y'(t)}{x'(t)} = \frac{3}{2}\frac{t^2-1}{t}$

vertical tangent $\Rightarrow$ $x'(t) = 0$ and $y'(t) \neq 0$. 1 1

1 $x'(t) = 0$ only when $t = 0$. $\quad y'(0) = 3(0^2-1) = -3 \neq 0$.

There is a vertical tangent line at

$(x(0), y(0)) = (-1, 0)$. 1

9. (a) Find the general solution of the differential equation

$$y' + xy = (x+1)e^x.$$

This is linear. The integrating factor is

(1) $\alpha(x) = e^{\int x\,dx} = e^{x^2/2}$ (1)

The solution is

(2)

$$y(x) = \frac{1}{\alpha(x)} \int \alpha(x) B(x)\,dx$$

$$= e^{-x^2/2} \int e^{x^2/2} \cdot (x+1)e^x\,dx \quad (1)$$

$$(1) \quad = e^{-x^2/2} \int (x+1) e^{x^2/2 + x}\,dx$$

$u = \frac{x^2}{2} + x$ (1)

$du = (x+1)\,dx$

$$= e^{-x^2/2} \int e^u\,du$$

$$= e^{-x^2/2}(e^u + C) = e^{-x^2/2}\left(e^{x^2/2 + x} + C\right)$$

$$= e^x + Ce^{-x^2/2} \quad (1)$$

(8)

(b) Solve the initial value problem

$$y' + 4xy^2 = 0, \qquad y(1) = -1.$$

(7)

This is separable.

$$y' = -4xy^2$$

$$(1) \quad \int \frac{dy}{y^2} = -4\int x\,dx$$

$$(1) \quad \frac{-1}{y} = -2x^2 + C \quad (1)$$

$$\frac{1}{y} = 2x^2 + C$$

$$(1) \quad y = \frac{1}{2x^2 + C}$$

Plug in initial condition:

$$(1) \quad -1 = \frac{1}{2(1)^2 + C} = \frac{1}{2 + C}$$

$$-1 = 2 + C$$

$$(1) \quad -3 = C$$

Solution is

$$(1) \quad \boxed{y = \frac{1}{2x^2 - 3}}$$

10. Consider the curve parametrized by

$$x(t) = 2t^2 + 6t + 5, \qquad y(t) = t^2 + 3t, \qquad 0 \le t \le 2.$$

(a) Find the speed of the parametrization at time t.

$\text{speed} = \sqrt{[x'(t)]^2 + [y'(t)]^2}$ 1

1 $x'(t) = 4t+6$ $\qquad y'(t) = 2t=3$ 1

4

$\text{speed} = \sqrt{(4t+6)^2 + (2t+3)^2} = \sqrt{4(2t+3)^2 + (2t+3)^2}$

$= \sqrt{5}(2t+3)$ $\qquad$ for $0 \le t \le 2.$

1

(b) Find a value for t where the speed is equal to 10.

$10 = \sqrt{5}(2t+3)$ 1

2

$2\sqrt{5} = 2t+3$

$\sqrt{5} - \frac{3}{2} = t$ 1

(c) Find the length of this curve.

$\text{length} = \int \text{speed}\, dt$ 1

4

$= \int_0^2 \sqrt{5}(2t+3)\,dt$ 1

$= \sqrt{5}\,(t^2+3t)\Big|_0^2$ 1

$= \sqrt{5}(4+6) = 10\sqrt{5}$ 1

11. Consider the power series
$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}(x-5)^n.$$

(a) Determine the radius of convergence of the power series above. Clearly indicate which tests you use, and verify that all necessary assumptions are satisfied.

1 Ratio Test   Assume $x \neq 5$, so terms nonzero

1 $$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(n+1)^2}{3^{n+1}}(x-5)^{n+1}}{\frac{(-1)^n n^2}{3^n}(x-5)^n} \right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{|x-5|}{3}$$ 1

$$= \frac{|x-5|}{3} \overset{?}{<} 1$$ 1 1

6

The limit is <1 when $|x-5|<3$, so the radius of convergence is $\boxed{3}$. 1

(b) Determine the interval of convergence of the power series above. Clearly indicate which tests you use, and verify that all necessary assumptions are satisfied.

It converges inside $(5-3, 5+3) = (2,8)$. Need to check the endpoints. 1

$x=2)$ $$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}(2-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}(-3)^n = \sum_{n=1}^{\infty} n^2$$
1

6

diverges by the Divergence Test. 1

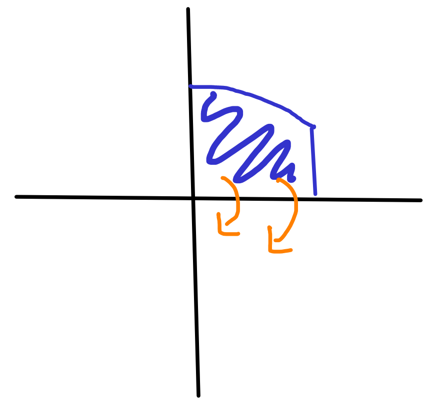
$x=8)$ $$\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}(8-5)^n = \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n} 3^n = \sum_{n=1}^{\infty} (-1)^n n^2$$
1

diverges by the Divergence Test. 1

The interval of convergence is $(2,8)$. 1

12. (a) Let R_1 be the region in the first quadrant bounded above by $f(x) = \dfrac{1}{1+x^2}$, on the sides by $x = 0$ and $x = 1$, and below by the x-axis.

Use the **Disk/Washer** method to find the volume of the solid obtained by rotating R_1 around the x-axis. Give an **exact value**, not a decimal approximation.



$$\text{Vol} = \pi \int f(x)^2\,dx \quad 2$$

$$= \pi \int_0^1 \frac{1}{(1+x^2)^2}\,dx \quad 1$$

$x = \tan\theta$ 1

$dx = \sec^2\theta$

$1 + x^2 = \sec^2\theta$

$$= \pi \int_0^{\pi/4} \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta \quad 2$$

$$= \pi \int_0^{\pi/4} \cos^2\theta\,d\theta \quad 1$$

$$= \pi \int_0^{\pi/4} \frac{1+\cos(2\theta)}{2}\,d\theta \quad 1$$

$$= \frac{\pi}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)\Big|_0^{\pi/4} \quad 1$$

$$= \frac{\pi}{2}\left[\left(\frac{\pi}{4} + \frac{1}{2}\right) - (0 + 0)\right]$$

$$= \frac{\pi}{8}(\pi + 2) \quad 1$$

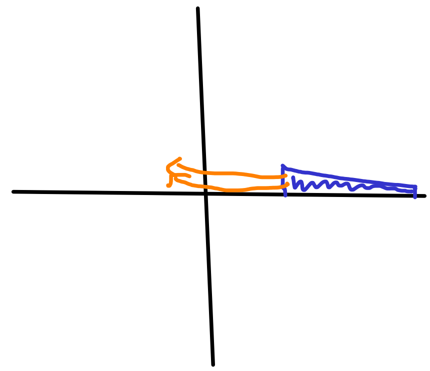
(b) Let R_2 be the region in the first quadrant bounded above by $f(x) = \dfrac{1}{(x+1)(x+2)}$, on the sides by $x = 1$ and $x = 3$, and below by the x-axis.

Use the **Shell** method to find the volume of the solid obtained by rotating R_2 around the y-axis. Give an **exact value**, not a decimal approximation.

2

$$\text{Vol} = 2\pi \int x\, f(x)\, dx$$

1 $$= 2\pi \int_1^3 \frac{x}{(x+1)(x+2)}\, dx$$



Partial Fractions: 1

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$ 1

$$x = A(x+2) + B(x+1)$$

Plug in $x=-1$: 1

$$-1 = A(-1+2) + 0$$
$$-1 = A$$ 1

Plug in $x=-2$:

$$-2 = 0 + B(-2+1)$$
$$-2 = -B, \quad B = 2$$ 1

$$\text{Vol} = 2\pi \int_1^3 \frac{-1}{x+1} + \frac{2}{x+2}\, dx$$

1 $$= 2\pi \Big[-\ln(x+1) + 2\ln(x+2) \Big]_1^3$$

$$= 2\pi \Big[(-\ln 4 + 2\ln 5) - (-\ln(2) + 2\ln(3)) \Big]$$

$$= 2\pi \ln \frac{2 \cdot 5^2}{3^2 \cdot 4} = 2\pi \ln \frac{25}{18}$$ 1