Exam 4

Name: ______________________ Section: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5"X11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Consider the integral $I = \int_0^2 \sqrt{x}\, dx$. Let R_n, L_n, and T_n, denote the right, left, and trapezoid rule estimates of I. Which of the following is true?

   A. $R_4 > I = T_4 > L_4$.

   B. $R_4 > T_4 > I > L_4$.

   C. $R_4 = T_4 = I > L_4$.

   D. $R_4 > I > T_4 > L_4$.

   E. $R_4 = I > T_4 > L_4$.

2. (5 points) Find the center of the ellipse with equation $y^2 + 3y + x^2 - 2x = 1$.

   A. $(\frac{3}{2}, -1)$

   B. $(1, -\frac{3}{2})$

   C. $(3, -1)$

   D. $(1, 3)$

   E. $(-1, -3)$

3. (5 points) Which of the following **sequences** converge?

   A. $b_n = \frac{2^n}{n!}$.

   B. $c_n = \frac{16n + (-1)^n}{n}$.

   C. $a_n = \ln(n^2 - 1) - \ln(n^2 + 1)$.

   D. None of the above.

   E. All of the above.

4. (5 points) The substitution $x = \sin(\theta)$ in the integral $\int \frac{dx}{x\sqrt{1-x^2}}$ leads to which of the following?

   A. $\int \frac{d\theta}{\sin(\theta)}$

   B. $\int \frac{\cos^2(\theta)d\theta}{\sin(\theta)}$

   C. $\int \frac{d\theta}{\cos(\theta)}$

   D. $\int \frac{d\theta}{\sec(\theta)}$

   E. $\int \frac{d\theta}{\sin^2(\theta)}$.

5. (5 points) Consider the curve C parametrized by $x(t) = t^3 - 3$ and $y(t) = t^2 + t - 1$. Find the slope of the tangent line to C at $(-2, 1)$.

   A. 3.

   B. $\frac{1}{3}$.

   C. $\frac{1}{9}$.

   D. 1.

   E. $\frac{2}{3}$.

6. (5 points) Evaluate $\int_0^\infty \frac{1}{(x+1)^2}\,dx$

   A. $\frac{1}{2}$.

   B. 1.

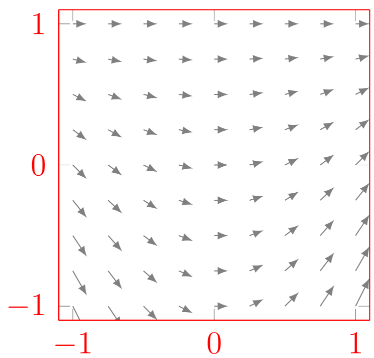
   C. 0.

   D. -1.

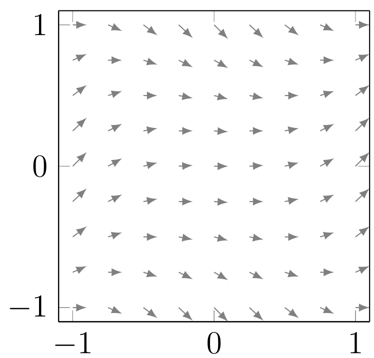
   E. This integral diverges.

7. (5 points) Which of the following is the direction field for the equation $y' = x(1-y)$?

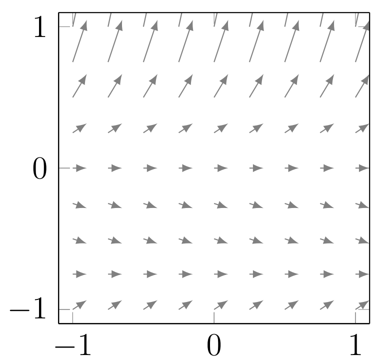
A.



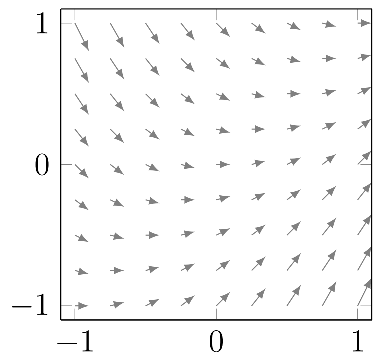
B.



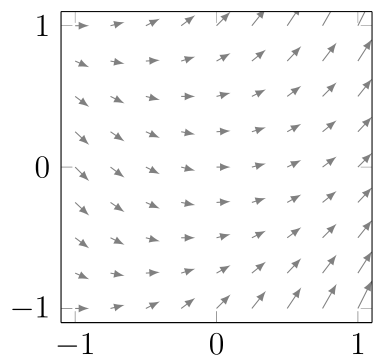
C.



D.



E.



8. (5 points) Find the center of mass of the system of particles given by a mass of 3 grams at $(-1,0)$, a mass of 5 grams at $(10,0)$, and a mass of 4 grams at $(0,6)$.

A. $(4,2)$.

B. $(2,4)$.

C. $(2, \frac{37}{12})$.

D. $(\frac{47}{12}, 2)$.

E. $(0, \frac{47}{12})$.

9. (5 points) Find the volume of a solid obtained by revolving the region between the graph of $f(x) = x(1-x)$ and the x-axis **around the y-axis.**

   A. $\frac{\pi^2}{6}$

   B. $\frac{1}{6}$

   C. $\frac{2\pi}{3}$

   D. $\frac{\pi}{4}$

   E. $\frac{\pi}{6}$

10. (5 points) A surface is created by rotating the graph of $f(x) = x + e^x$ from $x = 0$ to $x = 100$ around the x-axis. What is the integral that computes the area of this surface?

   A. $\int_0^{100} 2\pi(x+e^x)\sqrt{1+(1+e^x)^2}dx.$

   B. $\int_0^{100} 2\pi x(x+e^x)dx.$

   C. $\int_0^{100} \pi(x+e^x)^2dx.$

   D. $\int_0^{100} 2\pi x\sqrt{1+(1+e^x)^2}dx.$

   E. $\int_0^{100} 2x(x+e^x)^2dx.$

Free Response Questions

11. (a) (5 points) Compute $\int (x+1)e^x dx$.

Solution: Integration by parts: $u = x+1, dv = e^x dx$ gives $du = dx, v = e^x$:

$$\int (x+1)e^x dx = (x+1)e^x - \int e^x dx = (x+1)e^x - e^x + C = xe^x + C$$

(b) (5 points) Find the Maclaurin series for the function $\ln(1+x^2)$.

Solution: Start by taking a derivative $f'(x) = \dfrac{2x}{1+x^2}$.
Now use the expression for a geometric series:

$$f'(x) = 2x\sum_{n=0}^{\infty}(-x^2)^n = 2\sum_{n=0}^{\infty}(-1)^n x^{2n+1}$$

Now take an antiderivative:

$$\ln(1+x^2) = \int f'(x)dx = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+2}x^{2n+2} + C = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{2n+2} + C$$

Plug in $x = 0$ to determine that $0 = \ln(1) = 0 + C$ so that $C = 0$.

12. (10 points) Find the interval of convergence for the power series:

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{3^n}.$$

Solution: Apply the root test:

$$\sqrt[n]{\left|\frac{(2x-1)^n}{3^n}\right|} = \frac{|2x-1|}{3}$$

We get absolute convergence when $\frac{|2x-1|}{3} < 1$, which happens when $|x - \frac{1}{2}| < \frac{3}{2}$. This condition determines the open interval $(\frac{1}{2} - \frac{3}{2}, \frac{1}{2} + \frac{3}{2}) = (-1, 2)$. Now we test the endpoints:

$$\sum_{n=1}^{\infty} \frac{(2(-1)-1)^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n \frac{3^n}{3^n} = \sum_{n=1}^{\infty} (-1)^n,$$

$$\sum_{n=1}^{\infty} \frac{(2(2)-1)^n}{3^n} = \sum_{n=1}^{\infty} \frac{3^n}{3^n} = \sum_{n=1}^{\infty} 1.$$

Both of these series diverge, so we conclude that the interval of convergence is $(-1, 2)$.

13. (a) (5 points) Use Euler's method with **step size** $h = .1$ to estimate $y(.3)$ if y is a solution to the differential equation $y' = 2x(y+1)$, and $y(0) = 2$.

> **Solution:** The initial conditions are $x(0) = 0, y(0) = 2$, $h = .1$, and $F(x, y) = 2x(y + 1)$.
>
> $x(.1) = x(0) + h = 0 + .1 = .1$
> $y(.1) = y(0) + hF(x(0), y(0)) = 2 + (.1)F(0, 2) = 2 + .1(2(0)(2 + 1)) = 2$
>
> $x(.2) = x(.1) + h = .1 + .1 = .2$
> $y(.2) = y(.1) + hF(x(.1), y(.1)) = 2 + (.1)F(.1, 2) = 2 + .1(2(.1)(2 + 1)) = .06$
>
> $x(.3) = x(.2) + h = .2 + .1 = .3$
> $y(.3) = y(.2) + hF(x(.2), y(.2)) = .06 + (.1)F(.2, .06) =$
> $.06 + (.1)(2(.2)(.06 + 1)) = .0424$

(b) (5 points) Verify that $y(x) = 3e^{x^2} - 1$ is a solution to the differential equation $y' = 2x(y + 1)$ that satisfies $y(0) = 2$.

> **Solution:** We check $y' = (3e^{x^2} - 1)' = 3e^{x^2}(2x) = 2x(3e^{x^2} - 1 + 1) = 2x(y+1)$.
> Moreover, $y(0) = 3e^{0^2} - 1 = 3(1) - 1 = 2$.

14. (a) (5 points) The *cycloid* is the curve parametrized by the following functions:

$$x(\theta) = \theta - \sin(\theta),$$

$$y(\theta) = 1 - \cos(\theta).$$

Set up an integral which computes the arclength of the cycloid for $0 \leq \theta \leq 2\pi$.

Solution: The arclength function for a parametrized curve is:

$$\int \sqrt{(x'(\theta))^2 + (y'(\theta))^2}\ d\theta$$

We have $x'(\theta) = 1 - \cos(\theta)$, $y'(\theta) = \sin(\theta)$ so $(x'(\theta))^2 + (y'(\theta))^2 = (1-\cos(\theta))^2 + (\sin(\theta))^2 = 1 - 2\cos(\theta) + (\cos(\theta))^2 + (\sin(\theta))^2 = 2 - 2\cos(\theta)$. The arclength integral is:

$$\int_0^{2\pi} \sqrt{(2 - 2\cos(\theta)}\ d\theta$$

(b) (5 points) Find the slope of the line tangent to the polar curve $r = 2\sin(\theta)$ at the point defined by $\theta = \frac{\pi}{4}$.

Solution: The slope formula for a polar curve is given by $\dfrac{r'\sin(\theta) + r\cos(\theta}{r'\cos(\theta) - r\sin(\theta)}$.
In this case $r' = 2\cos(\theta)$ and $r = 2\sin(\theta)$, this gives:

$$\frac{2\cos(\theta)\sin(\theta) + 2\sin(\theta)\cos(\theta}{2\cos(\theta)\cos(\theta) - 2\sin(\theta)\sin(\theta)} = \frac{2\cos(\theta)\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}.$$

At $\frac{\pi}{4}$ the denominator is 0, so the tangent line is vertical.

15. (10 points) Use a partial fraction decomposition to compute $\int \frac{1}{(x-1)(x^2+1)} dx$.

Solution: The partial fraction decomposition is

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

so

$$1 = A(x^2+1) + (Bx+C)(x-1)$$

Collecting terms by degree on the left and right we get:

$$0x^2 = (A+B)x^2$$

$$0x = (C-B)x$$

$$1 = (A-C)$$

This gives $C = B = -A$ and $1 = -2C$ so $-\frac{1}{2} = C = B = -A$, and:

$$\frac{1}{(x-1)(x^2+1)} = \frac{1}{2}\frac{1}{(x-1)} - \frac{1}{2}\frac{x+1}{x^2+1}$$

Integrating, we get:

$$\frac{1}{2}\int \frac{1}{(x-1)}\, dx - \frac{1}{2}\int \frac{x+1}{x^2+1}\, dx =$$

$$\frac{1}{2}\int \frac{1}{(x-1)}\, dx - \frac{1}{2}\int \frac{x}{x^2+1}\, dx - \frac{1}{2}\int \frac{1}{x^2+1}\, dx =$$

$$\frac{1}{2}\ln(x-1) - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\arctan(x) + C$$