MA 114 Worksheet 08: Sequences

1. (a) Give the precise definition of a **sequence**.
   (b) What does it mean to say that $\lim_{x\to a} f(x) = L$ when $a = \infty$? Does this differ from $\lim_{n\to\infty} f(n) = L$? Why or why not?
   (c) What does it means for a sequence to converge? Explain your idea, not just the definition in the book.
   (d) Sequences can diverge in different ways. Describe two distinct ways that a sequence can diverge.
   (e) Give two examples of sequences which converge to 0 and two examples of sequences which converges to a given number $L \neq 0$.

2. Write the first four terms of the sequences with the following general terms:
   (a) $\frac{n!}{2^n}$
   (b) $\frac{n}{n+1}$
   (c) $(-1)^{n+1}$
   (d) $\{a_n\}_{n=1}^{\infty}$ where $a_n = \frac{3}{n}$.
   (e) $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2^{-n} + 2$.
   (f) $\{b_k\}_{k=1}^{\infty}$ where $b_k = \frac{(-1)^k)}{k^2}$.

3. Find a formula for the nth term of each sequence.
   (a) $\left\{\frac{1}{1}, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \ldots\right\}$
   (b) $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$
   (c) $\{1, 0, 1, 0, 1, 0, \ldots\}$
   (d) $\left\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots,\right\}$

4. Suppose that a sequence $\{a_n\}$ is bounded above and below. Does it converge? If not, find a counterexample.

5. The limit laws for sequences are the same as the limit laws for functions. Suppose you have sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} a_n = 15$, $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 1$. Use the limit laws of sequences to answer the following questions.
   (a) Does the sequence $\left\{\frac{a_n \cdot c_n}{b_n + 1}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?
   (b) Does the sequence $\left\{\frac{a_n + 3 \cdot c_n}{2 \cdot b_n + 2}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

6. (a) For what values of x does the sequence $\{x^n\}_{n=1}^{\infty}$ converge?
   (b) For what values of x does the sequence $\{n^x\}_{n=1}^{\infty}$ converge?
   (c) If $\lim_{n\to\infty} b_n = \sqrt{2}$, find $\lim_{n\to\infty} b_{n-3}$.

MA 114 Worksheet 09: Series

1. Write each of the following in summation notation.
   (a) $5 + \frac{5}{6} + \frac{5}{6^2} + \frac{5}{6^3} + \cdots$
   (b) $\frac{1}{1+1} + \frac{1}{2+3} + \frac{1}{2^2+3^2} + \frac{1}{2^3+3^3} + \cdots$
   (c) $1 + 3 + 5 + 7 + \cdots$

2. Identify the following statements as true or false and explain your answers.
   (a) If the sequence of partial sums of an infinite series is bounded the series converges.
   (b) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_n$ if the series converges.
   (c) $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$ if both series converge.
   (d) If c is a nonzero constant and if $\sum_{n=1}^{\infty} ca_n$ converges, then so does $\sum_{n=1}^{\infty} a_n$.
   (e) A finite number of terms of an infinite series may be changed without affecting whether or not the series converges.
   (f) Every infinite series with only finitely many nonzero terms converges.

3. Write the following in summation notation:
   (a) $\dfrac{1}{9} + \dfrac{1}{16} + \dfrac{1}{25} + \dfrac{1}{36} + \ldots$
   (b) $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

4. Calculate S_3, S_4, and S_5 and then find the sum of the telescoping series $S = \sum_{n=1}^{\infty} \left(\dfrac{1}{n+1} - \dfrac{1}{n+2}\right)$.

5. Use the formula for the sum of a geometric series to find the sum or state that the series diverges and why:
   (a) $\dfrac{1}{1} + \dfrac{1}{8} + \dfrac{1}{8^2} + \ldots$
   (b) $\sum_{n=0}^{\infty} \left(\dfrac{\pi}{e}\right)^n$

MA 114 Worksheet 10: Comparison Tests

1. (a) Explain the test for divergence. Why should you never use this test to prove that a series converges?
   (b) State the comparison test for series. Explain the idea behind this test.
   (c) Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy $0 \leq x_n \leq y_n$ for all n and that $\sum_{n=1}^{\infty} y_n$ is convergent. What can you conclude? What can you conclude if instead $\sum_{n=1}^{\infty} y_n$ diverges?
   (d) State the limit comparison test. Explain how you apply this test.

2. Use the appropriate test — Divergence Test, Comparison Test or Limit Comparison Test — to determine whether the infinite series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{2+5^n}$
   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{10}{n}\right)^{10}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2\sqrt{n}}$
   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{3n^2+14n+7}$
   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{2+5^n}$
   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+5n+2}$
   (j) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$
   (k) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2-\cos^2 n}$
   (l) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^4}$
   (m) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!}$

MA 114 Worksheet 11: Alternating Series, Absolute Convergence, & Conditional Convergence

1. (a) Let $a_n = \dfrac{n}{3n+1}$. Does $\{a_n\}$ converge? Does $\sum_{n=1}^{\infty} a_n$ converge?
   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ where $\lim_{n\to\infty} a_n = 0$.
   (c) Does there exist a convergent series $\sum_{n=1}^{\infty} a_n$ which satisfies $\lim_{n\to\infty} a_n \neq 0$? Explain.
   (d) When does a series converge absolutely? When does a series converge conditionally?
   (e) State the alternating series test.
   (f) State the Alternating Series Estimation Theorem.

2. Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

3. Test the following series for convergence or divergence.
   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$
   (b) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln(n)}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{2/3}}$
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n+5^n}$
   (e) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln(n)}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-5}{18}\right)$

4. Use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$
   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

5. Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}$ correct to four decimal places; *i.e.*, so that $|\text{error}| < 0.00005$.

MA 114 Worksheet 12: Ratio and Root Tests

1. (a) State the Root Test.
   (b) State the Ratio Test.

2. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.
   (a) To prove that the series $\sum_{n=1}^{\infty} a_n$ converges you should compute the limit $\lim_{n\to\infty} a_n$. If this limit is 0 then the series converges.
   (b) To apply the Ratio Test to the series $\sum_{n=1}^{\infty} a_n$ you should compute $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If this limit is less than 1 then the series converges absolutely.
   (c) To apply the Root Test to the series $\sum_{n=1}^{\infty} a_n$ you should compute $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If this limit is 1 or larger then the series diverges.
   (d) One way to prove that a series is convergent is to prove that it is absolutely convergent.
   (e) An infinite series converges when the sequence of partial sums converges.

3. Determine whether the series is absolutely convergent, conditionally convergent, or divergent. Remember that you may use **any** tests you have learned.
   (a) $\sum_{n=0}^{\infty} \left(\frac{3n^3+2n}{4n^3+1}\right)^n$
   (b) $\sum_{n=1}^{\infty} 13\cos(5)^{n-1}$
   (c) $\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$
   (d) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$
   (e) $\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}$
   (f) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$
   (g) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

MA 114 Worksheet 13: Power Series

1. (a) Give the definition of the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$
   (b) For what values of x does the series $\sum_{n=1}^{\infty} 2\big(\cos(x)\big)^{n-1}$ converge?
   (c) Find a formula for the coefficients c_k of the power series $\frac{1}{0!}+\frac{2}{1!}x+\frac{3}{2!}x^2+\frac{4}{3!}x^3+\cdots$.
   (d) Find a formula for the coefficients c_n of the power series $1+2x+x^2+2x^3+x^4+2x^5+x^6+\cdots$.
   (e) Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$ where $c \neq 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$.
   (f) Consider the function $f(x) = \frac{5}{1-x}$. Find a power series that is equal to $f(x)$ for every x satisfying $|x| < 1$.
   (g) Define the terms *power series*, *radius of convergence*, and *interval of convergence*.

2. Find the radius and interval of convergence for

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$

   (b) $4\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$

   (c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$

   (d) $\sum_{n=0}^{\infty} n!(x-2)^n$

   (e) $\sum_{n=0}^{\infty} (5x)^n$

   (f) $\sum_{n=0}^{\infty} \sqrt{n}x^n$

   (g) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

   (h) $\sum_{n=3}^{\infty} \frac{x^n}{3^n \ln n}$

   (i) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

   (j) $\sum_{n=4}^{\infty} \frac{(-1)^n x^n}{n^4}$

   (k) $\sum_{n=3}^{\infty} \frac{(5x)^n}{n^3}$

3. Use term-by-term integration and the fact that $\int_0^x \frac{1}{1+t^2}\,dt = \arctan(x)$ to derive a power series centered at $x=0$ for the arctangent function. HINT: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$.

4. Use the same idea as above to give a series expression for $\ln(1+x)$, given that $\int_0^x \frac{dt}{1+t} = \ln(1+x)$. You will again want to manipulate the fraction $\frac{1}{1+x} = \frac{1}{1-(-x)}$ as above.

5. Write $(1+x^2)^{-2}$ as a power series. HINT: use term-by-term differentiation.

MA 114 Worksheet 14: Taylor and Maclaurin Series

1. (a) Suppose that $f(x)$ has a power series representation for $|x| < R$. What is the general formula for the Maclaurin series for f?
   (b) Suppose that $f(x)$ has a power series representation for $|x - a| < R$. What is the general formula for the Taylor series for f about a?
   (c) Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$. Find the Maclaurin series for f.
   (d) Let $f(x) = 1 + 2x + 3x^2 + 4x^3$. Find the Taylor series for $f(x)$ centered at $x = 1$.

2. Assume that each of the following functions has a power series expansion. Find the Maclaurin series for each. Be sure to provide the domain on which the expansion is valid.
   (a) $f(x) = \ln(1 + x)$
   (b) $f(x) = xe^{2x}$

3. Use a known Maclaurin series to obtain the Maclaurin series for the given function. Specify the radius of convergence for the series.
   (a) $f(x) = \dfrac{x^2}{1 - 3x}$
   (b) $f(x) = e^x + e^{-x}$
   (c) $f(x) = e^{-x^2}$
   (d) $f(x) = x^5 \sin(3x^2)$
   (e) $f(x) = \sin^2 x$.
   Hint: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

4. Find the following Taylor expansions about $x = a$ for each of the following functions and their associated radii of convergence.
   (a) $f(x) = e^{5x}$, $a = 0$.
   (b) $f(x) = \sin(\pi x)$, $a = 1$.

5. Differentiate the series
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
to find a Taylor series for $\cos(x)$.

6. Use Maclaurin series to find the following limit: $\displaystyle\lim_{x\to 0} \frac{x - \tan^{-1}(x)}{x^3}$.

7. Approximate the following integral using a 6th order polynomial for $\cos(x)$.
$$\int_0^1 x\cos(x^3)\,dx$$

8. Use power series multiplication to find the first three terms of the Maclaurin series for
$$f(x) = e^x \ln(1 - x).$$

MA 114 Worksheet 15: Review for Exam 2

1. List the first five terms of the sequence:

   (a) $a_n = \dfrac{(-1)^n n}{n! + 1}$

   (b) $a_1 = 6$, $a_{n+1} = \dfrac{a_n}{n}$.

2. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = 3^n 7^{-n}$

   (b) $a_n = \dfrac{(-1)^{n+1} n}{n + \sqrt{n}}$

   (c) $a_n = \dfrac{\ln n}{\ln 2n}$

   (d) $a_n = \dfrac{\cos^2(n)}{2^n}$

3. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 2$.

4. Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{3^n}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{6 \cdot 2^{n-1}}{3^n}$

5. Determine whether the given series converges or diverges and state which test you used.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{7\sqrt{n}}{5n^{3/2} + 3n - 2}$

   (c) $\displaystyle\sum_{n=1}^{\infty} n! e^{-8n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\ln(n)}{5n+7}\right)^n$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{9n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

   (g) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan(n)$

6. Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{5n+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3+1}$

   (c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n^2}\right)$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

7. Find the radius and interval of convergence of each power series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{4^n n^4}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

8. Find a power series representation for each function and determine its radius of convergence.

(a) $f(x) = \dfrac{5}{1-4x^2}$

(b) $f(x) = \dfrac{x^2}{x^4+16}$

(c) $f(x) = \dfrac{3}{2+2x}$

(d) $f(x) = e^{-x^2}$

9. Using the formula

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$$

ind a power series for $\ln(1+x)$ and state its radius of convergence.

10. Use the Maclaurin series for $\cos(x)$ to compute

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2}.$$