

MA 114 Worksheet 16: Average Value of a Function

1. Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.
2. Find the average value of the following functions over the given interval.

(a) $f(x) = x^3, [0, 4]$

(b) $f(x) = \frac{2}{x}, [2, 6]$

(c) $f(x) = \cos(x), \left[0, \frac{\pi}{6}\right]$

(d) $f(x) = \frac{1}{x^2 + 1}, [-1, 1]$

(e) $f(x) = \frac{\sin(\pi/x)}{x^2}, [1, 2]$

(f) $f(x) = e^{-nx}, [-1, 1]$

(g) $f(x) = 2x^3 - 6x^2, [-1, 3]$

(h) $f(x) = x^n$ for $n \geq 0, [0, 1]$

3. In a certain city the temperature (in $^{\circ}F$) t hours after 9 am was modeled by the function $T(t) = 50 + 14 \sin\left(\frac{\pi t}{12}\right)$. Find the average temperature during the period from 9 am to 9 pm.
4. The velocity v of blood that flows in a blood vessel with radius R and length l at a distance r from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

where P is the pressure difference between the ends of the vessel and η is the viscosity of the blood. Find the average velocity (with respect to r) over the interval $0 < r < R$. Compare the average velocity with the maximum velocity.

5. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The function $f(t) = \frac{1}{4} \sin(2\pi t/5)$ can be used to model the rate of air flow into the lungs in L/s.
 - (a) Use this model to find the volume of inhaled air in the lungs at time t , assuming that the volume of air in the lungs at time $t = 0$ is 2.5 L
 - (b) Use this new function to compute the average volume of inhaled air in the lungs in one respiratory cycle.

MA 114 Worksheet 17: Volumes I

1. If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?
2. Calculate the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$, where cross sections perpendicular to the y -axis are squares.
3. The base of a solid is the triangle with vertices at $(10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y -axis are squares. Find the volume of the solid.
4. Calculate the volume of the solid whose base is a circle of radius r centered at the origin and which has square cross sections perpendicular to the x -axis.
5. Sketch the solid whose volume is given by the integral

$$\pi \int_0^1 (y^2 + 1)^2 - 1 \, dy.$$

6. Use disks or washers to find the volume of the solid obtained by rotating the given region about the specified line.
 - (a) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the x -axis.
 - (b) R is the region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.
 - (c) Compare the volumes you found in parts (a) and (b). Which is bigger? Why?
7. For each of the following, set up **but do not evaluate** an integral expression for the volume of the solid obtained by rotating the given region about the specified line.
 - (a) R is the region bounded by $y = \frac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$; about the x -axis.
 - (b) R is the region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the y -axis.
 - (c) R is the region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.
 - (d) R is the region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

MA 114 Worksheet 18: Volumes II

1. (a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the y -axis using the method of cylindrical shells.
(b) If you use the disk method to compute the same volume, are you integrating with respect to x or y ? Why?
(c) In general, how can we decide which method (disks/washers or cylindrical shells) to use?
2. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the y -axis.
3. Sketch the enclosed region and use the shell method to calculate the volume of the solid obtained by rotating the region about the y -axis.
 - (a) $y = 3x - 2$, $y = 6 - x$, $x = 0$
 - (b) $y = x^2$, $y = 8 - x^2$, $x = 0$, for $x \geq 0$
 - (c) $y = 8 - x^3$, $y = 8 - 4x$, for $x \geq 0$
4. (a) Use either method (disk/washer or cylindrical shell) to find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $x = 0$, and $y = 4$ about the y -axis.
(b) Which method do you think is easier to use in this case and why?
5. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \leq x \leq 1$ about the y -axis. Soda is extracted from the glass through a straw at the rate of $1/2$ cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Your answer should be implicitly in units of inches per second.)

MA 114 Worksheet 19: Arc Length and Surface Area

1.
 - (a) Write down the formula for the arc length of a function $f(x)$ over the interval $[a, b]$ including the required conditions on $f(x)$.
 - (b) Write down the formula for the surface area of a solid of revolution generated by rotating a function $f(x)$ over the interval $[a, b]$ around the x -axis. Include the required conditions on $f(x)$.
2. Find an integral expression for the arc length of the following curves. Do **not** evaluate the integrals.
 - (a) $f(x) = \sin(x)$ from $x = 0$ to $x = 2$
 - (b) $f(x) = x^4$ from $x = 2$ to $x = 6$
 - (c) $x^2 + y^2 = 1$
 - (d) $f(x) = e^x$ from $x = 0$ to $x = 1$
3. Find the arc length of the following curves.
 - (a) $f(x) = x^{3/2}$ from $x = 100$ to $x = 101$
 - (b) $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$
4. Set up a function $s(t)$ that gives the arc length of the curve $f(x) = 2x + 1$ from $x = 0$ to $x = t$. Find $s(4)$.
5. Compute the surface areas of revolution about the x -axis over the given interval for the following functions.
 - (a) $y = x$, $[0, 4]$
 - (b) $y = x^3$, $[0, 2]$
 - (c) $y = (4 - x^{2/3})^{3/2}$, $[0, 8]$
 - (d) $y = e^{-x}$, $[0, 1]$
 - (e) $y = \sqrt{4 - x^2}$, $[-1, 1]$
6. Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ about the x -axis.
7. Show that the surface area of a right circular cone of radius r and height h is $\pi r \sqrt{r^2 + h^2}$.
Hint: Rotate a line $y = mx$ about the x -axis for $0 \leq x \leq h$, where m is determined by the radius r .

MA 114 Worksheet 20: Centers of Mass

1. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 2)$, $(2, -1)$, and $(4, 0)$ respectively.
2. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3, 0)$, $(b, 0)$, and $(0, 6)$, where $b > 3$. Find the center of mass.
3. Consider the region under the graph of $y = 1 - x^2$ for $0 \leq x \leq 1$.
 - (a) Given uniform density $\rho = 1$, compute the mass M and the moments M_x and M_y .
 - (b) Given uniform density $\rho = 2$, compute the mass M and the moments M_x and M_y .
 - (c) Use the previous information to compute the centroid of the region in both cases.
 - (d) Does the centroid change if ρ changes?
4. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \leq x \leq 4$.
5. Find the centroid of the region between $f(x) = x - 1$ and $g(x) = 2 - x$ for $3/2 \leq x \leq 2$.

MA 114 Worksheet 21: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?
(b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?
(c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?
(d) Do the two sets of parametric equations

$$y_1(t) = 5 \sin(t), \quad x_1(t) = 5 \cos(t), \quad 0 \leq t \leq 2\pi$$

and

$$y_2(t) = 5 \sin(t), \quad x_2(t) = 5 \cos(t), \quad 0 \leq t \leq 20\pi$$

represent the same parametric curve? Explain.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, (\frac{t}{\pi})^2)$, for $0 \leq t \leq 2\pi$.
(a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$.
(b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?
(c) Use the above information to plot the curve.
3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.
(a) $x = \sqrt{t}$, $y = 1 - t$ for $t \geq 0$.
(b) $x = 3t - 5$, $y = 2t + 1$ for $t \in \mathbb{R}$.
(c) $x = \cos(t)$, $y = \sin(t)$ for $t \in [0, 2\pi]$.
4. Represent each of the following curves as parametric equations traced just once on the indicated interval.
(a) $y = x^3$ from $x = 0$ to $x = 2$.
(b) $\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1$.
5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds at a constant speed. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

MA 114 Worksheet 22: Review for Exam 3

1. Find the volume of the following solids.
 - (a) The solid obtained by rotating the region bounded by $y = x^2$ and $x = y^2$ about the x -axis,
 - (b) The solid obtained by rotating the region bounded by $x = y^2$ and $x = 1$ about the line $x = 1$,
 - (c) The solid obtained by rotating the region bounded by $y = 4x - x^2$ and $y = 3$ about the line $x = 1$,
 - (d) The solid with circular base of radius 1 and cross-sections perpendicular to the base that are equilateral triangles.
2. Find the surface area of the surface of revolution obtained by rotating the given curve about the given axis.
 - (a) $y = \sqrt{x+1}$, $0 \leq x \leq 3$; about x -axis,
 - (b) $x = 3t^2$, $y = 2t^3$, $0 \leq t \leq 5$; about y -axis.
3. Compute the arc length of the following curves.
 - (a) $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $0 \leq \theta \leq 2\pi$,
 - (b) $y = \sqrt{2-x^2}$, $0 \leq x \leq 1$.
4. Find the centroid of the region bounded by $y = \sqrt{x}$ and $y = x$.
5. Find the average value of the function $y = 3 \sin(x) + \cos(2x)$ on the interval $[0, \pi]$.
6. Compute the slope of the tangent line to the curve in Problem 3(a) above, with $a = 8$, at the point $(1, 3^{3/2})$. Use this to determine an equation for the tangent line.