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MA 123

Review Worksheet #0: Algebra Review

Before diving into our study of calculus, let's first review some topics in algebra that we may use this semester.

Key Points

1. A strong foundation in algebra is necessary to be successful in calculus.

Domain

Example 1. Determine the domain of the following functions:

(a) $f(x) = \dfrac{5x^2 - 13x - 6}{x - 3}$

(b) $g(x) = \sqrt{9 - x}$

Evaluating/Graphing Functions

Example 2. Use the functions f and g from Example 1 to compute the following values:

(a) $f(3)$

(b) $f(5)$

(c) $f(10)$

(d) $g(3)$

(e) $g(5)$

(f) $g(10)$

A *piecewise function* can be thought of as a function defined by several different rules. In order to evaluate a piecewise function, we first need to determine which rule applies.

Example 3. Let

$$h(x) = \begin{cases} 2x+7 & \text{if } x \le -2, \\ 3 & \text{if } -2 < x \le 1, \\ -x+1 & \text{if } x > 1. \end{cases}$$

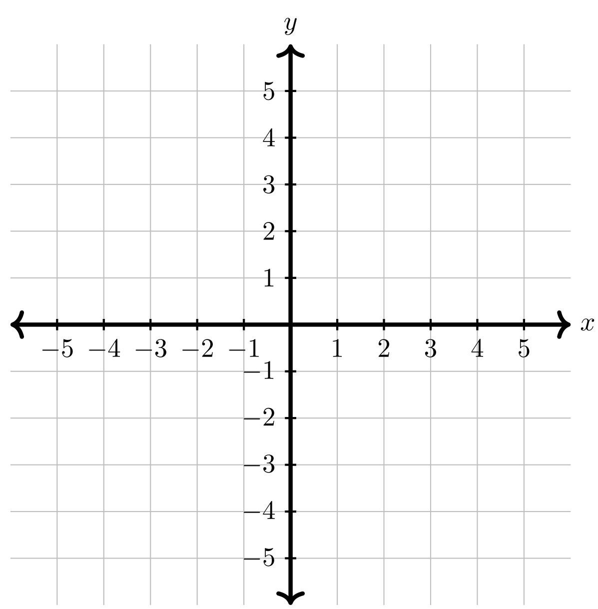
Compute the following values:

(a) $h(-4)$

(b) $h(1)$

(c) $h(3)$

Example 4. Graph the piecewise function from Example 3.



Exponents

Below is a list of exponent rules that may be needed this semester.

1. $a^{1/n} = \sqrt[n]{a}$
2. $a^0 = 1$
3. $a^{-n} = \dfrac{1}{a^n}$
4. $a^m a^n = a^{m+n}$
5. $\dfrac{a^m}{a^n} = a^{m-n}$
6. $(a^m)^n = a^{mn}$
7. $(ab)^n = a^n b^n$
8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Example 5. Simplify the expression $\dfrac{x^6(2x)^3}{x^{15}}$ so that the answer is in the form cx^n.

Example 6. Write the expression $\sqrt[3]{x^8}$ in the form x^n.

Factoring/Quadratic Formula

Example 7. Solve $x^2 + 4x - 32 = 0$.

(a) By factoring the quadratic expression.

(b) By using the quadratic formula.

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Review Worksheet #1: Average Rate of Change

As we saw in Introductory Worksheet #1, average rate of change measures the average rate at which a function changes over a given interval.

Key Points

1. average speed $= \dfrac{\text{distance traveled}}{\text{time elasped}}$
2. The average rate of change of a function $f(x)$ between $x = x_1$ and $x = x_2$ is given by
$$\text{average rate of change} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$
3. The average rate of change of a function $f(x)$ over a given interval $[x_1, x_2]$ is equal to the slope of the secant line between $x = x_1$ and $x = x_2$ on the graph of $y = f(x)$.

Example 8. A train leaves city A at 8:00am and arrives at city B at 10:00am. The average speed of the train from A to B was 60 mph. The train leaves city B at 10:00am and arrives at city C at 1:00pm. Determine the average speed of the train from B to C, given that the average speed from A to C was 50 mph.

Example 9. Let $f(x) = \sqrt{1+2x}$. Determine the average rate of change of $f(x)$ from $x = 4$ to $x = 12$.

Example 10. Let $g(x) = 4x^2 + 3$. Determine the average rate of change of $g(x)$ on the interval $[x, x+h]$.

Example 11. Let $h(t) = 5t^2$. Determine a value of t such that the average rate of change of $h(t)$ from 2 to t equals 50.

Example 12. Let $p(x) = \frac{1}{x}$. Determine a value of x such that the average rate of change of $p(x)$ from 1 to x equals $-\frac{1}{4}$.

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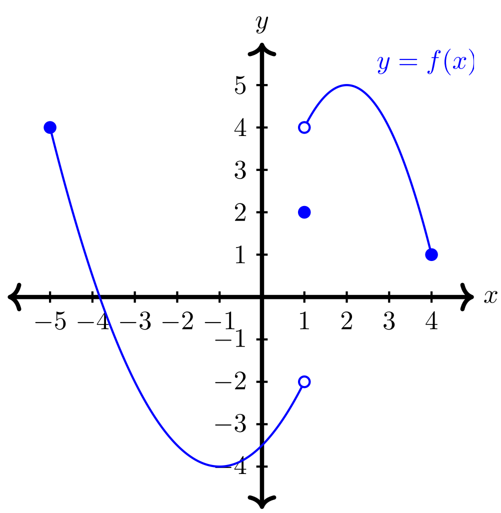
Review Worksheet #2: Limits and Continuity

As we saw in Introductory Worksheet #2, limits tell us what is happening to the value of a function as the independent variable gets closer and closer to a specified value. These limits can be determined via a graph of the function or, if the function is "nice enough", by plugging the specified value into the function. In some cases, a limit may fail to exist or we may have to do some additional work before direct substitution works.

Key Points

1. The limit $\lim_{x\to c^-} f(x)$ is the value that $f(x)$ approaches as x gets closer to c for values of x less than c.
2. The limit $\lim_{x\to c^+} f(x)$ is the value that $f(x)$ approaches as x gets closer to c for values of x greater than c.
3. The limit $\lim_{x\to c} f(x)$ exists if both $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ exist and are equal.
4. (new) The *end behavior* of a rational function can be determined by considering only the highest order terms of the polynomials in the numerator and denominator.
5. (new) A function $f(x)$ is *continuous* at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

Example 13. Below is the graph of a piecewise function $f(x)$.



Use the graph to evaluate the following limits/values.

(a) $\lim_{x\to 1^-} f(x)$

(b) $\lim_{x\to 1^+} f(x)$

(c) $\lim_{x\to 1} f(x)$

(d) $f(1)$

Example 14. Evaluate $\lim_{x \to 1} \left[(x^2 + 3x - 2)(x - 4) \right]$.

Example 15. Evaluate $\lim_{x \to 2} \frac{x^3 - 5}{x + 4}$.

Below are some properties of limits:

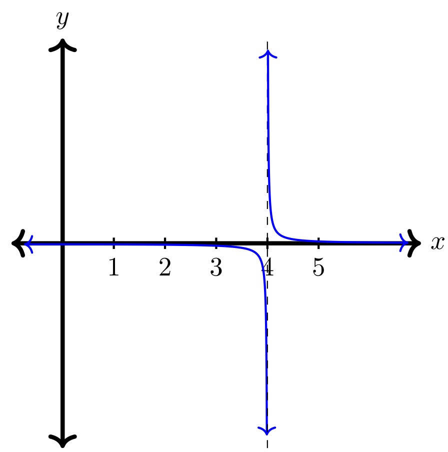
- $\lim_{x \to c} (f \pm g)(x) = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$
- $\lim_{x \to c} (f \cdot g)(x) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
- $\lim_{x \to c} \left(\frac{f}{g} \right)(x) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ provided $\lim_{x \to c} g(x) \neq 0$
- $\lim_{x \to c} \big(k \cdot f(x) \big) = k \cdot \lim_{x \to c} f(x)$ for any constant k

Example 16. Suppose $\lim_{x \to -1} f(x) = 4$ and $\lim_{x \to -1} g(x) = 7$. Evaluate $\lim_{x \to -1} \big[3f(x) + g(x)(x + 5) \big]$.

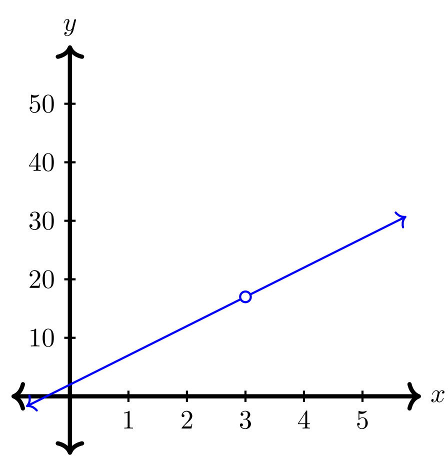
The next few examples involve limits in which direct substitution yields division by zero. There are two cases to consider:

- If the numerator results in a nonzero value, then the *limit does not exist* since the values of the function near $x = c$ are getting arbitrarily large.
- If the numerator is also zero, then we have to do more work to determine whether the limit exists.

Example 17. Evaluate $\lim_{x \to 4} \dfrac{1}{x^2 - 16}$.



Example 18. Evaluate $\lim_{x \to 3} \dfrac{5x^2 - 13x - 6}{x - 3}$. *Hint:* $5x^2 - 13x - 6 = (5x + 2)(x - 3)$.



Example 19. Evaluate $\lim_{x \to 4} \frac{x^2 - x - 12}{x^2 - 16}$.

Example 20. Evaluate $\lim_{x \to 0^+} \frac{7\sqrt{x}}{x}$.

Example 21. Evaluate $\lim_{x \to 0} \left(\frac{4}{x} - \frac{6x + 4}{x} \right)$.

Example 22. Evaluate $\lim_{h \to 0} \frac{|h|}{h}$.

We can also consider the end behavior of a function by considering limits as $x \to \pm\infty$. A function f is called a *rational function* if $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are both polynomials. For polynomials, the term with the highest power of x dominates the other terms when x is very large. Then, when determining the limit of a rational function $f(x) = \frac{p(x)}{q(x)}$ as $x \to \pm\infty$, we only need to consider the highest order term of $p(x)$ and $q(x)$.

Example 23. Evaluate $\lim_{x\to\infty} \frac{6x^2 + 9x - 2}{6x + 3x^2 + 7}$.

Example 24. Evaluate $\lim_{x\to-\infty} \frac{(5x+3)^2}{4x^2 + 7x + 1}$.

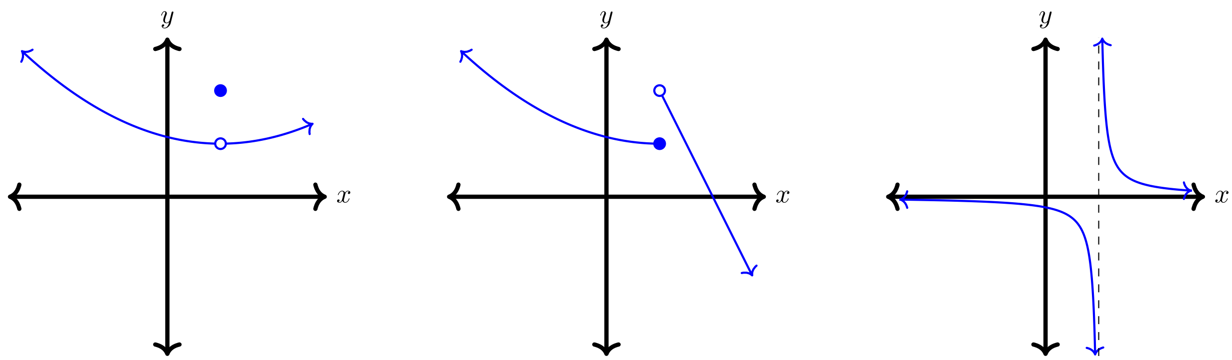
Example 25. Evaluate $\lim_{x\to\infty} \frac{2x^4 + 1}{x^3 - 8}$.

Example 26. Evaluate $\lim_{x\to\infty} \frac{x^3 - 8}{2x^4 + 1}$.

You may have heard the term "continuous function" prior to taking this class. A function f is **continuous at $x = c$** if

$$\lim_{x\to c} f(x) = f(c).$$

A function f is **continuous on an interval** if it is continuous at every point of that interval. This means that the graph of f has no holes, jumps, or vertical asymptotes at any point in the interval. Thus, you can draw the graph of f from one end of the interval to the other without lifting your pencil off the paper. Below are examples of *discontinuities*.



Hole **Jump** **Asymptote**

Example 27. Let

$$f(x) = \begin{cases} Ax^2 & \text{if } x < 2, \\ 1 - Ax & \text{if } x \geq 2. \end{cases}$$

Find a value of A such that $f(x)$ is continuous at $x = 2$.

Note that polynomial functions are continuous. Below are a few facts about continuous functions. If f and g are continuous functions at $x = c$ and k is any real number, then

- $(f \pm g)(x)$ is continuous at $x = c$,
- $(f \cdot g)(x)$ is continuous at $x = c$,
- $\left(\dfrac{f}{g}\right)(x)$ is continuous at $x = c$ when $g(c) \neq 0$, and
- $k \cdot f(x)$ is continuous at $x = c$.

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Review Worksheet #3: Instantaneous Rate of Change and The Derivative

As we saw in Introductory Worksheet #3, instantaneous rate of change measures the rate at which a function changes at a specific point.

Key Points

1. The instantaneous rate of change of a function $f(x)$ at $x = x_1$ is given by

$$\text{instantaneous rate of change} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}.$$

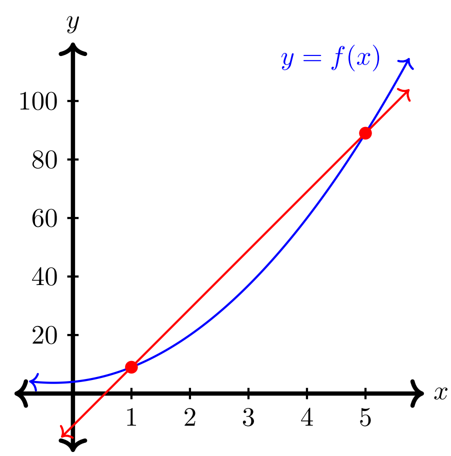
2. The instantaneous rate of change of a function $f(x)$ at $x = x_1$ is equal to the slope of the tangent line to the graph of $y = f(x)$ at $x = x_1$.

3. The derivative of a function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

4. The derivative of a function $f(x)$ evaluated at $x = x_1$ yields the instantaneous rate of change of f at $x = x_1$.

Example 28. Let $f(x) = 3x^2 + 2x + 4$. Determine a value of c between 1 and 5 such that the average rate of change of $f(x)$ from $x = 1$ to $x = 5$ is equal to the instantaneous rate of change of $f(x)$ at $x = c$.



Example 29. Let $f(x) = mx+b$ be an arbitrary linear function. Use the definition of the derivative to show that $f'(x) = m$.

Example 30. Let $g(x) = ax^2 + bx + c$ be an arbitrary quadratic function. Use the definition of the derivative to show that $g'(x) = 2ax + b$.

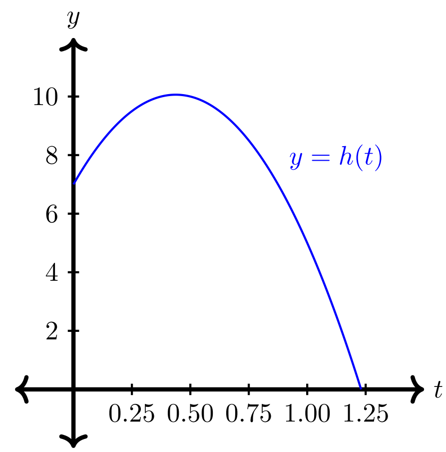
Example 31. Let $p(x) = 6x^2 - 4x + 1$. Determine a value of d such that $p'(d) = 8$.

Example 32. Let $q(x) = -x^2 + 4x + 12$.

(a) Determine the value of x for which the tangent line to the graph of $q(x)$ has slope -6.

(b) Determine the value of x for which the tangent line to the graph of $q(x)$ has slope 0.

Example 33. If $h(t) = -16t^2 + 14t + 7$ represents the height of an object at time t, determine the height of the object when its speed is zero (prior to hitting the ground).



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Review Worksheet #4: Tangent Lines and Differentiability

As we saw in Introductory Worksheet #4, an equation of the tangent line to the graph of a function $f(x)$ at $x = x_0$ can be determined from the values of $f(x_0)$ and $f'(x_0)$. We also saw that a function is not differentiable wherever it is not continuous or it has a corner point.

Key Points

1. An equation of the tangent line to the graph of a function $y = f(x)$ at $x = x_0$ is

$$y = f(x_0) + f'(x_0)(x - x_0).$$

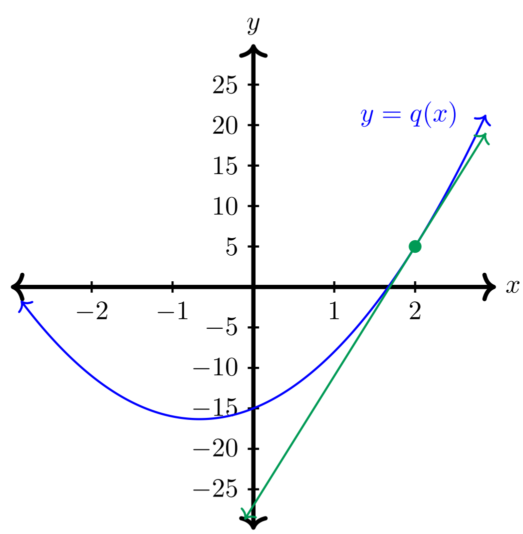
2. A function is not differentiable wherever the graph of the function has a hole, a jump, a vertical asymptote, or a corner point.

Example 34. Suppose the equation of the tangent line to the graph of $y = p(x)$ at $x = 4$ is

$$y = 1 + 9(x - 4).$$

Determine the values of $p(4)$ and $p'(4)$.

Example 35. Determine the slope-intercept form of the equation of the tangent line to the graph of $q(x) = 3x^2 + 4x - 15$ at $x = 2$.



Example 36. Use the definition of the derivative to determine the derivative of $g(x) = \sqrt{x} + a$.

Example 37. Suppose that the equation of the tangent line to the graph of the function

$$g(x) = \sqrt{x} + a$$

at $x = 1$ is

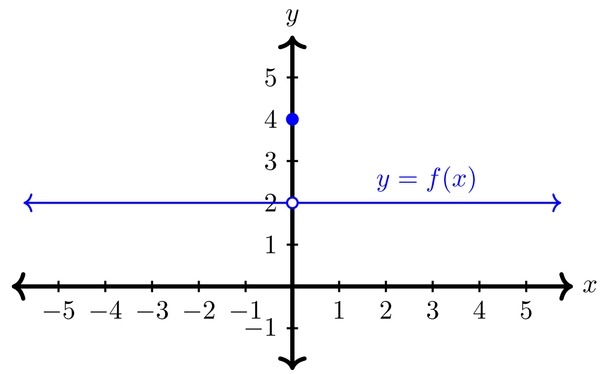
$$y = mx + 2.$$

Determine the values of a and m.

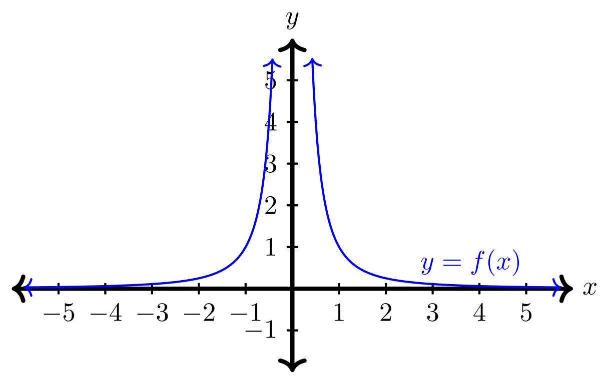
We saw in Introductory Worksheet #4 (see Practice 16) that a function is not differentiable wherever it has a jump or a corner point. The next example shows that a function is not differentiable wherever it has a hole or a vertical asymptote.

Example 38. Show that each of the following functions are not differentiable at $x = 0$.

(a) $f(x) = \begin{cases} 2 & \text{if } x \neq 0, \\ 4 & \text{if } x = 0 \end{cases}$



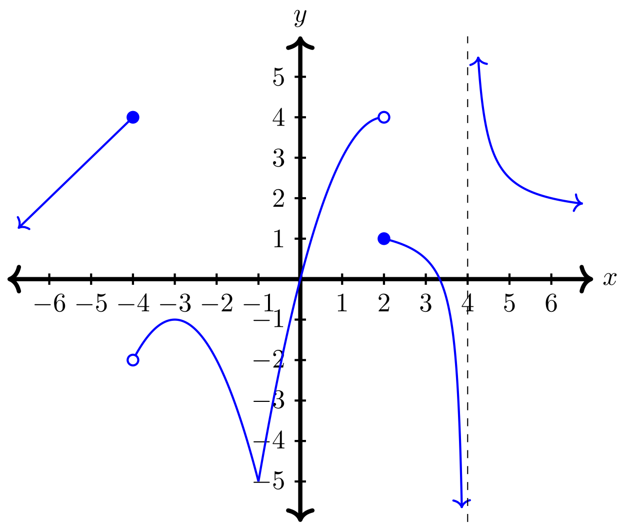
(b) $f(x) = \dfrac{1}{x^2}$



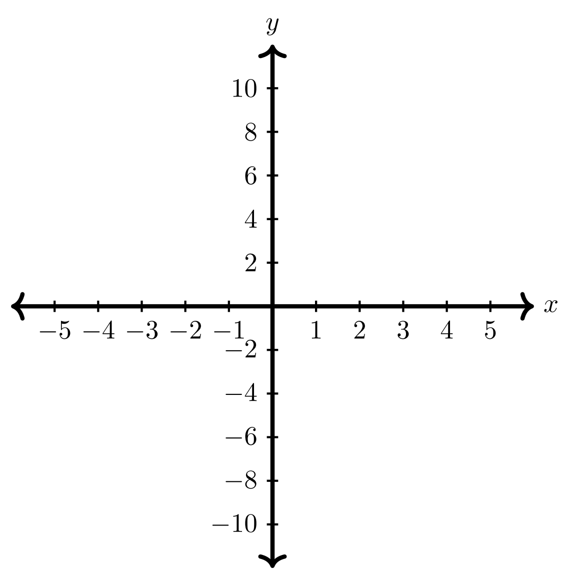
The table below summarizes what to look for in a graph to determine points at which a function is not continuous/differentiable.

<u>Not Continuous</u> <u>Not Differentiable</u>

Example 39. Below is the graph of a piecewise function $f(x)$. Determine all values of x for which the function is not continuous and all values of x for which the function is not differentiable.



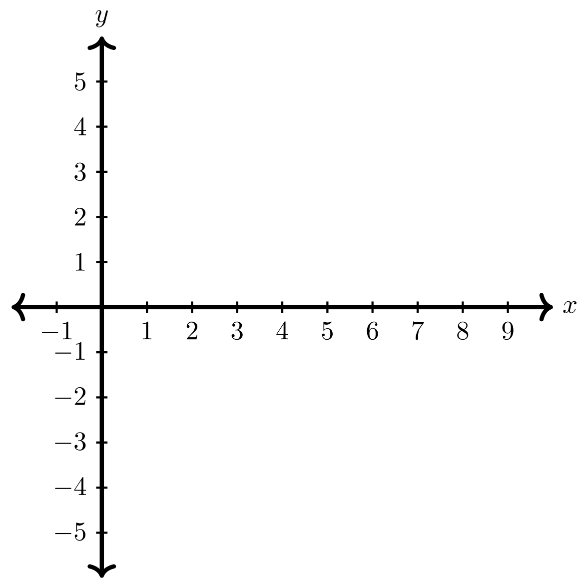
Example 40. Determine all values of x for which $g(x) = \left|x^2 - 4x - 5\right|$ is not differentiable.



Example 41. Determine all values of x where the function

$$h(x) = \begin{cases} x - 5 & \text{if } x < 5, \\ 5 - x & \text{if } 5 \le x \le 8, \\ \dfrac{x^2}{4} - 15 & \text{if } x > 8 \end{cases}$$

is continuous but not differentiable.

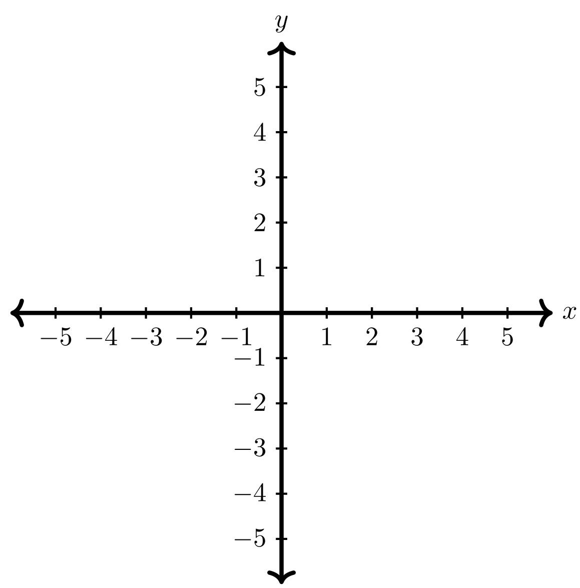


Example 42. Sketch the graph of a function $y = f(x)$ which satisfies the following properties:

(1) $\lim_{x \to \pm\infty} f(x) = 4$ (2) $\lim_{x \to -2^-} f(x) = 2$ (3) $\lim_{x \to -2^+} f(x) = 1$ (4) $\lim_{x \to 1^-} f(x) = 3$

(5) $\lim_{x \to 1^+} f(x) = 3$ (6) $f(-2) = -1$ (7) $f(1) = 3$

(8) $f(x)$ is continuous everywhere except at $x = -2$



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Review Worksheet #5: Formulas for Derivatives

As we saw in Introductory Worksheet #5, there are formulas for determining the derivative of certain types of functions.

Key Points

1. The derivative of a constant function $f(x) = c$ is $f'(x) = 0$.
2. The derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$.
3. Let c be a constant and $f(x)$ be a differentiable function. Then $(cf(x))' = cf'(x)$.
4. Let $f(x)$ and $g(x)$ be differentiable functions. Then $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.
5. Let $f(x)$ and $g(x)$ be differentiable functions. Then
$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$
6. (new) Let $f(x)$ and $g(x)$ be differentiable functions. Then
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$
7. (new) Let $f(x)$ and $g(x)$ be functions, with f differentiable at x and g differentiable at $f(x)$. Then $(g(f(x)))' = f'(x)g'(f(x))$.

The Constant Rule

Example 43. Let $f(x) = 5$. Determine $f'(x)$.

The Power Rule

Example 44. Determine the derivative of each of the following functions.

(a) $f(x) = \sqrt[3]{x^5}$

(b) $f(x) = \dfrac{1}{\sqrt[7]{x}}$

The Constant Multiple Rule

Example 45. Let $f(x) = 3\sqrt{x}$. Determine $f'(x)$.

The Sum/Difference Rule

Example 46. Determine the derivative of each of the following functions.

(a) $f(x) = x^5 + 3x^2 - \sqrt{x} + 17$

(b) $f(x) = \dfrac{x^9 + x^2}{x^4}$

Example 47. Determine the equation of the tangent line to $f(x) = 4x^2 - 7x^3$ at $x = 1$.

The Product Rule

Example 48. Determine the derivative of each of the following functions.
(a) $f(x) = (2x + 1)(4x^3 + 5)$

(b) $f(x) = (x - 3)(x + 9)(x - 5)$

Example 49. Suppose $f(x) = x^2 + 3x - 7$, $g(1) = 3$, and $g'(1) = -8$. If $h(x) = f(x)g(x)$, determine $h'(1)$.

The Quotient Rule

Let $f(x)$ and $g(x)$ be differentiable functions. Then $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

Example 50. Let $f(x) = \dfrac{2x+1}{4x^3+5}$. Determine $f'(x)$.

Example 51. Suppose $f(x) = 4x + 8$, $g(2) = 3$, and $g'(2) = -2$. If $h(x) = \dfrac{f(x)}{g(x)}$, determine $h'(2)$.

The Chain Rule

A function $h(x)$ is said to be a composite function of $f(x)$ followed by $g(x)$ if

$$h(x) = (g \circ f)(x) = g(f(x)).$$

For instance, consider the function $h(x) = (5x - 7)^4$. To evaluate $h(2)$, we would first compute $5(2) - 7 = 3$ and then $3^4 = 81$. That is, $h(x) = g(f(x))$ where $f(x) = 5x - 7$ and $g(x) = x^4$.

Example 52. Determine functions $f(x)$ and $g(x)$ such that $h(x) = g(f(x))$.

(a) $h(x) = (3x^2 + 10x + 2)^5$

(b) $h(x) = \dfrac{1}{\sqrt[3]{2x - 5}}$

Let $f(x)$ and $g(x)$ be functions, with f differentiable at x and g differentiable at $f(x)$. Then $(g(f(x)))' = f'(x)g'(f(x))$.

Example 53. Determine the derivative of each of the following functions.

(a) $h(x) = (3x^2 + 10x + 2)^5$

(b) $h(x) = \dfrac{1}{\sqrt[3]{2x - 5}}$

Example 54. Suppose $g(x) = \sqrt{f(x)}$ and the equation of the tangent line to $f(x)$ at $x = 1$ is $y = 4 + 2(x - 1)$. Determine $g'(1)$.

Example 55. Suppose $g(f(x)) = x^2$ and $f'(1) = 5$. Determine $g'(f(1))$.

Example 56. Suppose $h(x) = f(x)g(2x)$. If $f(1) = 3$, $f'(1) = 4$, $g(2) = 5$ and $g'(2) = 6$, determine $h'(1)$.

Higher Order Derivatives

We've seen how to determine the derivative of certain types of functions. Provided the derivative is itself differentiable, we can take the derivative of the derivative. Namely, if $f(x)$ and $f'(x)$ are differentiable, then

$$f''(x) = (f'(x))'$$

is the second derivative of $f(x)$. Similarly, we can define even higher order derivatives of $f(x)$ if they exist, and often use the notation $f^{(3)}(x)$, $f^{(4)}(x)$,..., to denote the third derivative, fourth derivative, and so on.

Example 57. Let $f(x) = x^5$.

(a) Determine $f'(x)$.

(b) Determine $f''(x)$.

(c) Determine $f^{(3)}(x)$.

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Review Worksheet #6: Exponential and Logarithmic Functions

As we saw in Introductory Worksheet #6, there are formulas for determining the derivative of natural exponential and logarithmic functions.

Key Points

1. The derivative of $f(x) = e^x$ is $f'(x) = e^x$.
2. The derivative of $g(x) = \ln(x)$ is $g'(x) = \dfrac{1}{x}$.

Example 58. Let $f(x) = e^{5x}$.
(a) Determine $f'(x)$.

(b) Determine $f''(x)$.

(c) Determine $f^{(10)}(x)$.

Example 59. Let $f(x) = 3e^x - 6x^e$. Determine $f'(x)$.

Example 60. Determine the slope of the tangent line to the graph of $f(x) = xe^x$ at $x = 4$.

Example 61. Let $f(x) = (2x + 6)e^{7x+8}$. Determine $f'(x)$.

Example 62. Let $f(x) = e^{\sqrt{5x+7}}$. Determine $f'(x)$.

Example 63. Let $g(x) = \ln(x^5 + 1)$. Determine $g'(x)$.

Example 64. Let $g(x) = \ln(\ln(\ln(x)))$. Determine $g'(x)$.

Example 65. Suppose $g(x) = \ln(\ln(\ln(f(x))))$. If $f(1) = A$ and $f'(1) = B$, determine $g'(1)$.

We conclude this worksheet by considering an application of exponential growth and decay. Let $P(t)$ denote the amount of a quantity as a function of time. We say that $P(t)$ *grows exponentially* as a function of time if

$$P(t) = P_0 e^{rt},$$

where P_0 and r are positive constants that depend on the specific problem. At time $t = 0$, we have

$$P(0) = P_0 e^{r \cdot 0} = P_0 e^0 = P_0.$$

Hence, P_0 denotes the initial amount of the quantity at time $t = 0$. Note that in the case of exponential growth,

$$P'(t) = P_0(re^{rt}) = r(P_0 e^{rt}) = rP(t).$$

That is, the rate of growth of a quantity that grows exponentially is proportional to the quantity present (r is called the proportionality constant).

Some quantities *decay exponentially*. In this case we have $P(t) = P_0 e^{-rt}$, where again P_0 and r are positive constants. Note that in the case of exponential decay,

$$P'(t) = P_0(-re^{-rt}) = -r(P_0 e^{-rt}) = -rP(t).$$

That is, the rate of growth of a quantity that decays exponentially is proportional to the quantity present ($-r$ is the proportionality constant here).

Example 66. A bacteria culture starts with 5000 bacteria and the population doubles after 6 hours. Determine the number of bacteria after 9 hours if the number of bacteria present after t hours is given by $y(t) = P_0 e^{rt}$.

Example 67. If the bacteria in a culture triples in 2 hours, how many hours will it take before 7 times the original number of bacteria is present? Again use $y(t) = P_0 e^{rt}$ for the number of bacteria present after t hours.

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Review Worksheet #7: Applications of the Derivative

As we saw in Introductory Worksheet #7, derivatives are used to measure rates of change in the real world.

Key Points

1. Derivatives can be used to approximate the additional cost, revenue, and profit from producing and selling one more item.

Example 68. The height of a sand dune (in centimeters) is represented by $h(t) = 500 - 10t^2$ cm, where t is measured in years since 1995. Determine the values of $h(6)$ and $h'(6)$, including units.

Example 69. A company that makes thing-a-ma-bobs has a start up cost of \$23,312. It costs the company \$2.92 to make each thing-a-ma-bob. The company charges \$3.93 for each thing-a-ma-bob.

(a) Determine the cost function for this company if x thing-a-ma-bobs are produced.

(b) Determine the revenue function for this company if x thing-a-ma-bobs are sold.

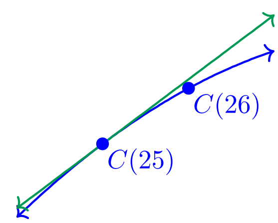
(c) What is the minimum number of thing-a-ma-bobs that the company must produce and sell to make a profit?

Example 70. The total cost (in dollars) of producing x items is

$$C(x) = 2500 + 40x - 0.7x^2.$$

(a) Approximate the cost of the 26th item using the marginal cost function.

(b) Determine the exact cost of producing the 26th item.



(c) Determine the average cost per item if 25 items are produced.

(d) Determine the marginal average cost at a production level of 25 items.

(e) Use parts (c) and (d) to estimate the average cost per item if 26 items are produced.

(f) Determine the exact average cost per item if 26 items are produced.

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Review Worksheet #8: Extreme Value Theorem

As we saw in Introductory Worksheet #8, the derivative can be used to determine the minimum and maximum values of a continuous function on a closed interval.

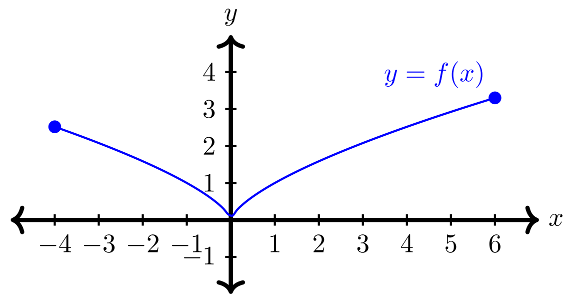
Key Points

1. If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains a minimum and maximum value at an endpoint a or b, at an interior point where $f'(x) = 0$, or at an interior point where $f'(x)$ does not exist.

Example 71. Determine the minimum and maximum values of $f(x) = 1 + x + x^2 + x^3 + x^4 + x^5$ on the interval $[0, 1]$.

Example 72. Determine the minimum and maximum values of $g(x) = \dfrac{x+4}{x+11}$ on the interval $[1, 8]$.

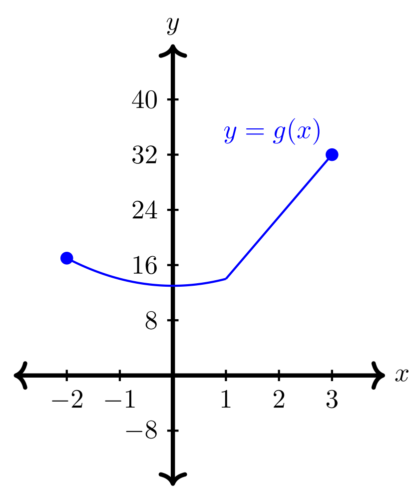
Example 73. Determine the minimum and maximum values of $f(x) = \sqrt[3]{x^2}$ on the interval $[-4, 6]$.



Example 74. Determine the minimum and maximum values of

$$g(x) = \begin{cases} x^2 + 13 & \text{if } x \leq 1, \\ 9x + 5 & \text{if } x > 1 \end{cases}$$

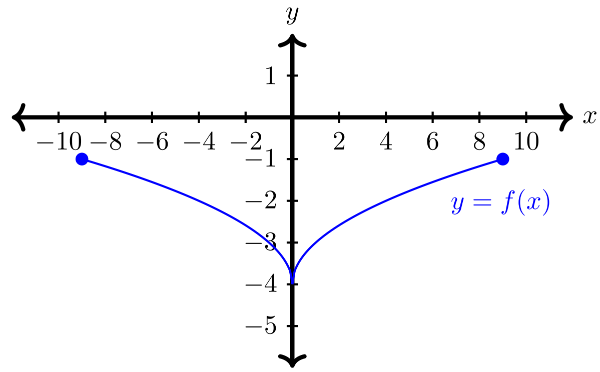
on the interval $[-2, 3]$.



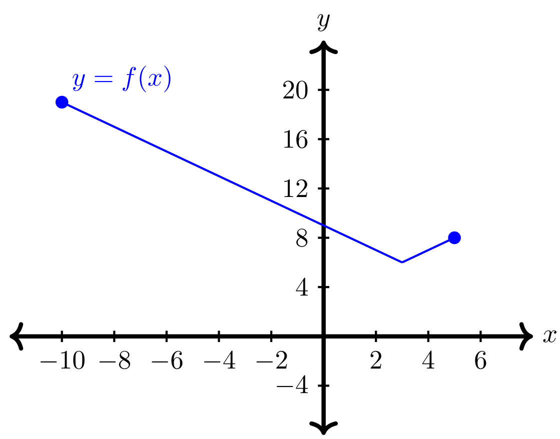
Example 75. Determine the minimum and maximum values of

$$f(x) = \begin{cases} -4 + \sqrt{x} & \text{if } x > 0, \\ -4 + \sqrt{-x} & \text{if } x \leq 0 \end{cases}$$

on the interval $[-9, 9]$.



Example 76. Determine the minimum and maximum values of $g(x) = |x - 3| + 6$ on the interval $[-10, 5]$.



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Review Worksheet #9: Critical Values and Increasing/Decreasing

As we saw in Introductory Worksheet #9, the first derivative provides useful information about the behavior of a function. Namely, the first derivative can be used to determine on which intervals a function is increasing or decreasing.

Key Points

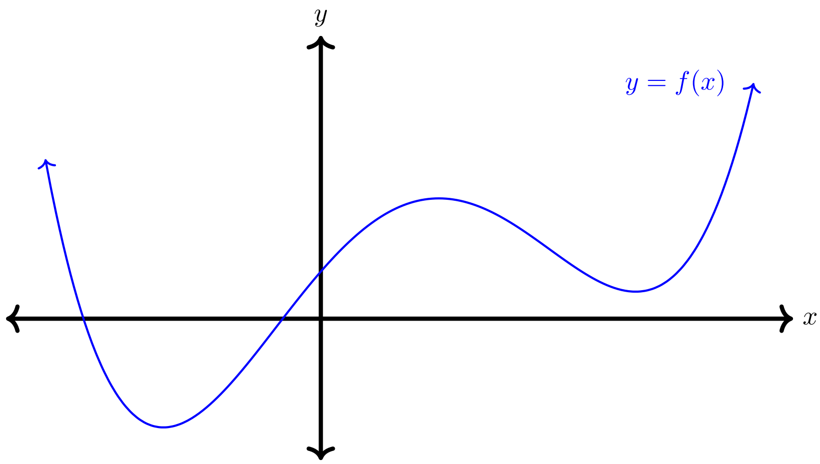
1. If f is differentiable on an interval I and $f'(x) > 0$ for all $x \in I$, then f is *increasing* on I.
2. If f is differentiable on an interval I and $f'(x) < 0$ for all $x \in I$, then f is *decreasing* on I.

Example 77. Let $f(x) = x^3 + 2x^2 - 4x + 9$. Determine the intervals over which f is increasing/decreasing.

Example 78. Determine the smallest value of A such that the function $g(x) = x^4 - 25x^2 + 144$ is increasing for all x in the interval $(A, 0)$.

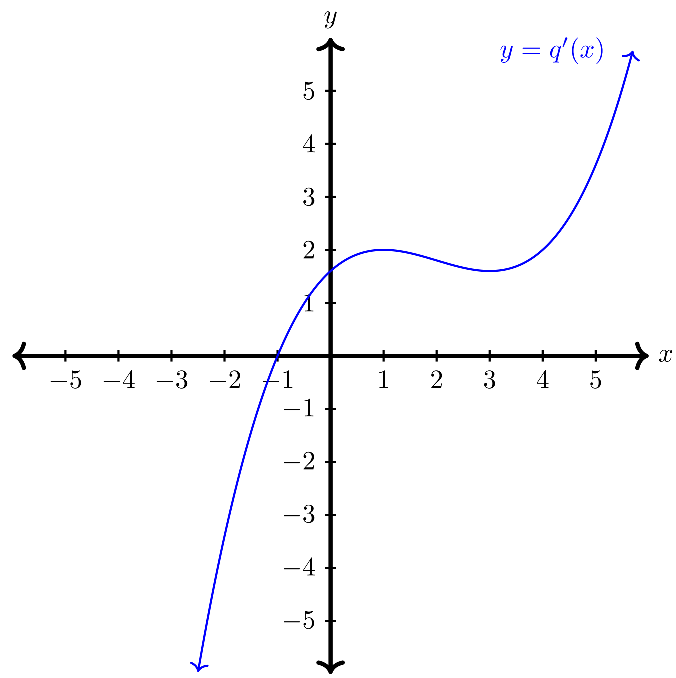
Example 79. Suppose $h'(x) = -\dfrac{7}{(x-10)^2}$. For which value of x does $h(x)$ attain its maximum value over the interval $[-9, 8]$?

An extreme value can either be classified as a absolute (a.k.a global) maximum/minimum or a relative (a.k.a local) maximum/minimum.



Example 80. Suppose $p(x) = \dfrac{e^{3x}}{x}$. Find the x values in the interval $(0, \infty)$ where p attains its absolute minimum value.

Example 81. The graph of $q'(x)$ is given below. Determine the intervals over which q is increasing/decreasing.



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Review Worksheet #10: Concavity and Curve Sketching

As we saw in Introductory Worksheet #10, the second derivative provides useful information about the behavior of a function. Namely, the second derivative can be used to determine on which intervals the graph of a function is concave up or concave down.

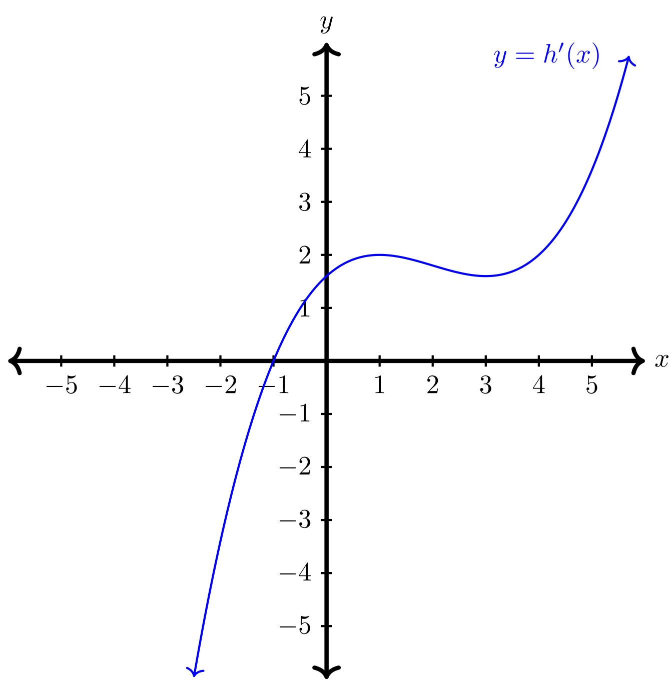
Key Points

1. The graph of $y = f(x)$ is *concave up* on an interval I if $f''(x) > 0$ for all $x \in I$.
2. The graph of $y = f(x)$ is *concave down* on an interval I if $f''(x) < 0$ for all $x \in I$.
3. A point on the graph of $y = f(x)$ at which the concavity changes is called an *inflection point*.

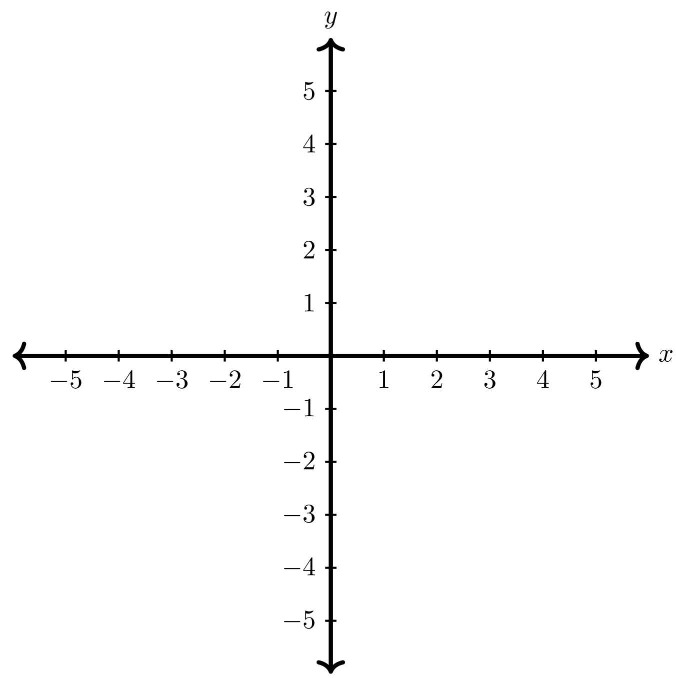
Example 82. Let $f(x) = \dfrac{1}{x^2 - 2x + 1}$. Determine the intervals over which the graph of f is concave up/concave down. Are there any inflection points?

Example 83. Let $g(x) = xe^{5x}$. Determine the intervals over which the graph of g is concave up/concave down. Are there any inflection points?

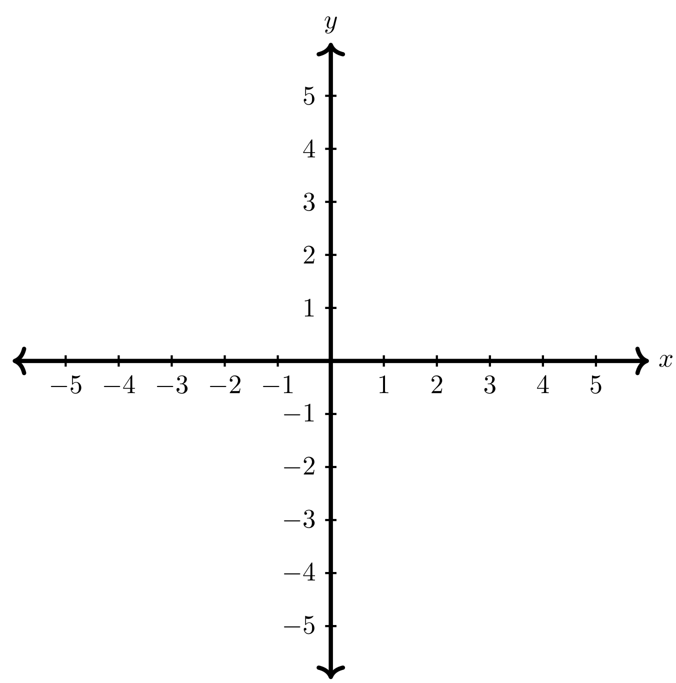
Example 84. The graph of $h'(x)$ is given below. Determine the intervals over which the graph of h is concave up/concave down. Are there any inflection points?



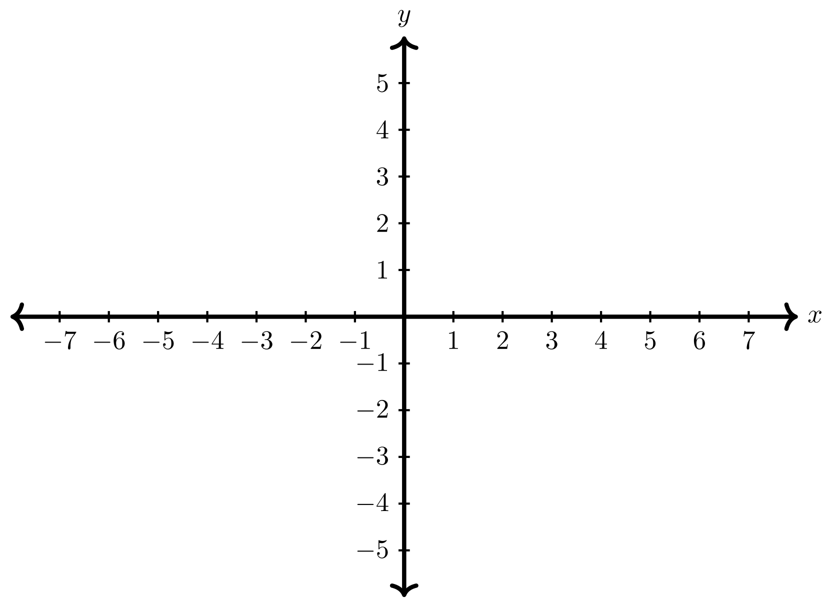
Example 85. Sketch the graph of a continuous function $y = f(x)$ satisfying $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > -1$, $f''(x) > 0$ for $x < 4$, and $f''(x) < 0$ for $x > 4$.



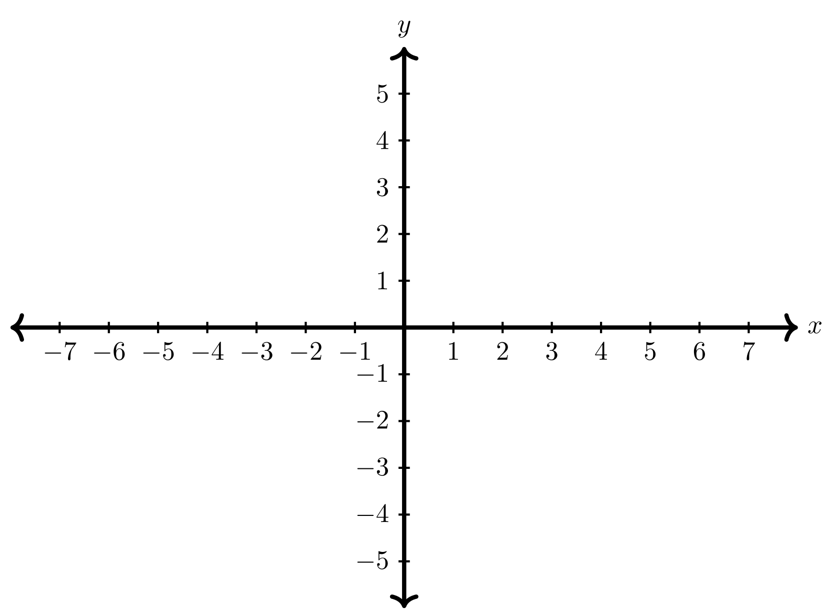
Example 86. Sketch the graph of a continuous function $y = g(x)$ satisfying $g(2) = 4$, $g'(x) < 0$ for $x > 2$, $g'(x) > 0$ for $x < 2$, and $g''(x) > 0$ for $x \in (-\infty, 2) \cup (2, \infty)$.



Example 87. Sketch the graph of a continuous function $y = h(x)$ satisfying $h(2) = 1$, $h'(x) > 0$ for $x \in (-\infty, -3) \cup (2, 5)$, $h'(x) < 0$ for $x \in (-3, 2) \cup (5, \infty)$, $h''(x) > 0$ for $x \in (-\infty, -3)$, and $h''(x) < 0$ for $x \in (-3, 2) \cup (2, \infty)$.



Example 88. Sketch the graph of a continuous function $y = p(x)$ satisfying $p'(x) > 0$ for $x \in (-\infty, -2) \cup (4, \infty)$, $p'(x) < 0$ for $x \in (-2, 4)$, $p''(x) > 0$ for $x \in (1, \infty)$, and $p''(x) < 0$ for $x \in (-\infty, 1)$.



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Review Worksheet #11: Optimization

As we saw in Introductory Worksheet #11, the first derivative can be used to determine the optimal value of a quantity given a set of constraints.

Key Points

1. The steps for solving an optimization problem are:

   Step 1. Identify the variables.

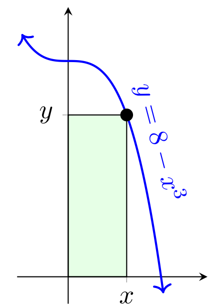
   Step 2. State the *objective function*, as well as whether we are trying to maximize or minimize this quantity.

   Step 3. Determine any constraints on the variables and solve for one variable in terms of the other. Are there any bounds on the independent variable?

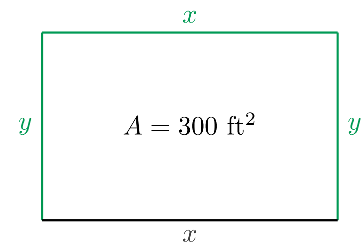
   Step 4. Rewrite the objective function from Step 2 as a function of one variable, including the domain.

   Step 5. Determine the critical values of the objective function from Step 4 and perform a number line test to answer the question.

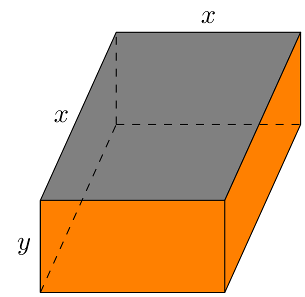
Example 89. Determine the dimensions of the rectangle of largest area which can be inscribed inside the region bounded by the positive x-axis, the positive y-axis, and the graph of $y = 8 - x^3$.



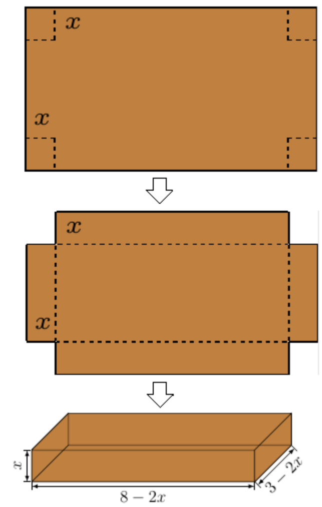
Example 90. A landscape architect plans to enclose a 300 ft^2 rectangular region in a garden. He will use shrubbery costing \$25 per foot along three sides and fencing costing \$10 per foot along the fourth side. Determine the dimensions of the garden that yield the minimum total cost.



Example 91. A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$5 per square foot and the metal for the sides costs \$10 per square foot. Determine the dimensions that minimize cost if the box has a volume of 20 ft^3.



Example 92. An open box is to be made out of a 3-foot by 8-foot piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. Determine the dimensions of the resulting box that has the largest volume.



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Review Worksheet #12: The Idea of the Integral

As we saw in Introductory Worksheet #12, definite integrals evaluate the signed area between the graph of a given function and the horizontal axis (x-axis).

Key Points

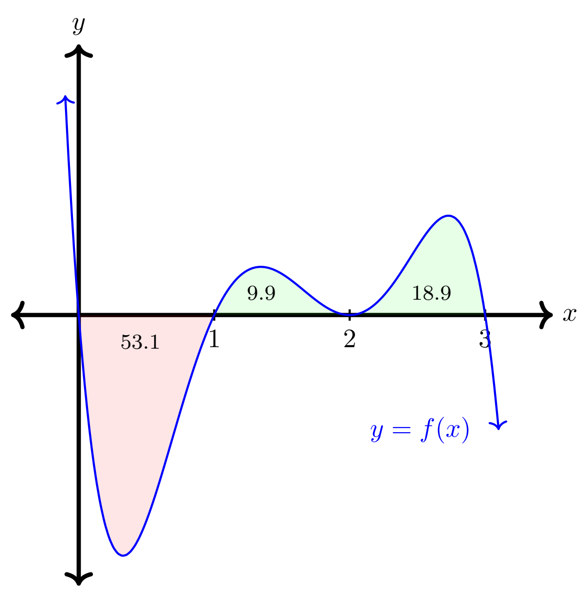
1. Any area above the horizontal axis is considered a positive area and any area below the horizontal axis is considered a negative area.

Below are some useful properties of definite integrals:

- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
- $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$
- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Example 93. Evaluate the following definite integrals using the graph of $f(x)$.

(a) $\int_0^2 f(x)\,dx$



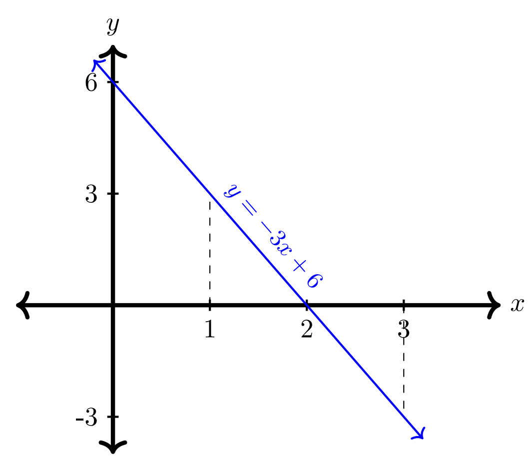
(b) $\int_0^1 2f(x)\,dx$

(c) $\int_3^2 f(x)\,dx$

Example 94. Use the graph to evaluate the following definite integrals.

(a) $\int_0^1 (-3x+6)\,dx$

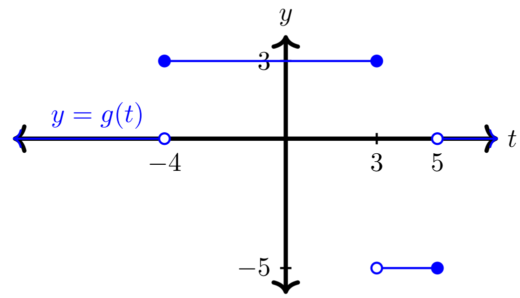
(b) $\int_0^3 (-3x+6)\,dx$



(c) $\int_0^1 (-9x+18)\,dx$

Example 95. Let $h(x) = \int_{-4}^{x} g(t)\,dt$ where

$$g(t) = \begin{cases} 0 & \text{if } t < -4, \\ 3 & \text{if } -4 \le t \le 3, \\ -5 & \text{if } 3 < t \le 5, \\ 0 & \text{if } t > 5. \end{cases}$$



Determine the following values.

(a) $h(-7)$

(b) $h(-3)$

(c) $h(4)$

(d) At which x value does $h(x)$ attain its maximum?

Another important property of definite integrals is that

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Example 96. If $\int_0^2 f(x)\,dx = 12$, $\int_0^3 f(x)\,dx = -2$, $\int_0^2 g(x)\,dx = 13$ and $\int_2^3 g(x)\,dx = -11$, evaluate the following definite integrals.

(a) $\int_0^2 (f(x) + g(x))\,dx$

(b) $\int_0^3 (f(x) + g(x))\,dx$

(c) $\int_2^3 (3f(x) + 2g(x))\,dx$

(d) Suppose $5a \int_0^3 f(x)\,dx + \int_0^3 g(x)\,dx = 0$. Determine the value of a.

Example 97. If $\int_{-3}^{8} f(x)\,dx = 24$, $\int_{-3}^{-1} f(x)\,dx = 4$ and $\int_{5}^{8} f(x)\,dx = 7$, evaluate the following definite integrals.

(a) $\int_{-1}^{5} f(x)\,dx$

(b) $\int_{5}^{-1} (6f(x) - 8)\,dx$

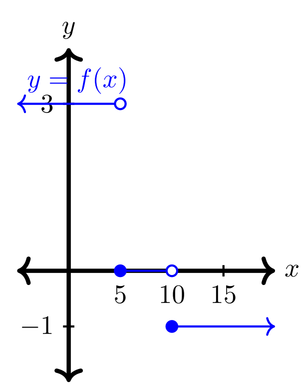
Definite integrals can be used to determine the average value of a function over an interval. Namely, the **average value** of $f(x)$ from $x = a$ to $x = b$ is

$$AV(f) = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx.$$

Example 98. Let

$$f(x) = \begin{cases} 3 & \text{if } x < 5, \\ 0 & \text{if } 5 \leq x < 10, \\ -1 & \text{if } x \geq 10. \end{cases}$$

Determine the average value of $f(x)$ from $x = 0$ to $x = 15$.



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Review Worksheet #13: Formulas for Antiderivatives and u-Substitution

As we saw in Introductory Worksheet #13, there are known rules for determining the antiderivative of functions of the form $f(x) = x^n$ or $f(x) = e^x$. Furthermore, a technique known as u-substitution can be used to determine an antiderivative when the function inside the integral is a result of a chain rule.

Key Points

1. **The Power Rule** ($n \neq -1$): Let $n \neq -1$. The antiderivative of $f(x) = x^n$ is $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$.
2. $n = -1$ **case**: The antiderivative of $f(x) = \frac{1}{x}$ is $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.
3. **The Exponential Rule**: The antiderivative of $f(x) = e^x$ is $\displaystyle\int e^x\,dx = e^x + C$.
4. u-**Substitution** is used when the function inside the integral is a result of a chain rule.

Example 99. Determine the following antiderivatives.

(a) $\displaystyle\int \left(\sqrt[5]{x} + \frac{1}{x^2}\right) dx$

(b) $\displaystyle\int \frac{x+8}{x}\,dx$

(c) $\displaystyle\int 3e^x\,dx$

Example 100. Determine the following antiderivatives.

(a) $\int e^{7x}\,dx$

(b) $\int \frac{4\ln|x|+1}{x}\,dx$

(c) $\int \frac{3x^2}{x^3+8}\,dx$

(d) $\int \frac{x^3+8}{3x^2}\,dx$

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Review Worksheet #14: Fundamental Theorem of Calculus (Part I)

As seen in Introductory Worksheet #14, the first part of the Fundamental Theorem of Calculus shows how to take the derivative of a definite integral of the form $\int_a^x f(t)\,dt$.

Key Points

1. The Fundamental Theorem of Calculus (Part I) states $\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$, where $f(t)$ is a continuous function on an interval $[a,b]$ and $x \in (a,b)$.
2. The above result can be generalized as follows: $\frac{d}{dx}\left(\int_a^{g(x)} f(t)\,dt\right) = g'(x)\cdot f(g(x))$.

Example 101. Let $F(x) = \int_{-2}^{x} (t^4 + t^3 + t + 5)\,dt$. Determine $F'(x)$.

Example 102. Let $G(x) = \int_{\sqrt{x}}^{1} \frac{t^2}{5+3t^4}\,dt$. Determine $G'(x)$.

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Review Worksheet #15: Fundamental Theorem of Calculus (Part II)

As we saw in Introductory Worksheet #15, the second part of the Fundamental Theorem of Calculus shows how to evaluate a definite integral using antiderivatives.

Key Points

1. The Fundamental Theorem of Calculus (Part II) states $\int_a^b f(x)\,dx = F(b) - F(a)$, where $F(x)$ is *any* antiderivative of $f(x)$ on $[a, b]$.

Example 103. Evaluate $\displaystyle\int_2^5 \frac{x^2+11}{x^2}\,dx$.

Example 104. Evaluate $\displaystyle\int_{-1}^6 |x|\,dx$.

Example 105. Evaluate $\int_0^{10} f(x)\,dx$, where $f(x) = \begin{cases} 6x & \text{if } x < 7, \\ \dfrac{9}{x} & \text{if } x \geq 7. \end{cases}$

Example 106. Determine the average value of $f(x) = x^2$ on the interval $[4, 8]$.

Example 107. Evaluate $\int_1^x \frac{t^2+1}{(6t^3+18t+1)^6}\,dt$.

Example 108. Evaluate $\int_0^T \frac{2x}{x^2+1}\,dx$.

Let $v(t)$ be the *velocity* of an object at time t.

1. The total *displacement* of the object over the time interval $[a, b]$ is given by

$$\int_a^b v(t)\, dt.$$

2. The total *distance traveled* by the object over the time interval $[a, b]$ is given by

$$\int_a^b |v(t)|\, dt.$$

Example 109. A train travels at a velocity of $v(t) = 66t$ mph for the 1^{st} half-hour and then at a constant velocity of $v(t) = 33$ mph (t is measured in hours and distance is measured in miles). How far did the train travel during the 1^{st} hour?

Example 110. A rock is dropped from a height of 40 feet. Its velocity at time t is given by $v(t) = -33t$ ft/s (t is measured in seconds and distance is measured in feet).

(a) How far did the rock travel during the 1^{st} second?

(b) How far from the ground is the rock after 1 second?

(c) How long does it take for the rock to hit the ground?