MA123, Chapter10: Idea of the Integral SOLUTIONS

Chapter Goals:

- Understand the relationship between the area under a curve and the definite integral.
- Understand the relationship between velocity (speed), distance and the definite integral.
- Use the definite integral to compute the average value of a function over an interval

Assignments: Assignment 18 Assignment 19

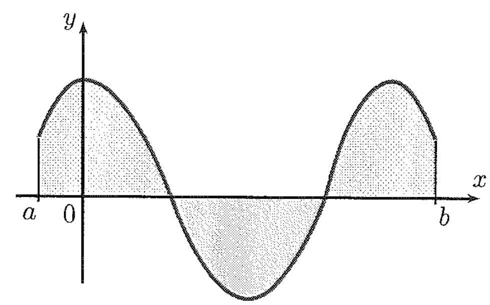
▶ **The basic idea:** Definite integrals compute *signed area.*

Definition: The **definite integral**

$$\int_a^b f(x)\,dx$$

computes the signed area between the graph of $y = f(x)$ and the x-axis on the interval $[a, b]$.

- If $a < b$ and the region is above the x-axis, the area has positive sign.
- If $a < b$ and the region is below the x-axis, the area has negative sign.
- If the function takes on both positive and negative values on $[a, b]$, the "positive" and "negative" areas will cancel out.



That is, if $a < b$, then

$$\boxed{\int_a^b f(x)\,dx = [\text{area of the region(s) lying above the } x\text{-axis}] - [\text{area of the region(s) lying below the } x\text{-axis}]}$$

Notation: Given $\int_a^b f(x)\,dx$, we call $f(x)$ the **integrand**, dx identifies x as the variable, and a and b are called the **limits of integration**.

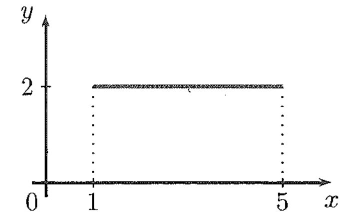
Applications: Suppose $v(t)$ measures the velocity of an object at time t.

(a) $\int_a^b v(t)\,dt$ measures the displacement of the object from $t = a$ to $t = b$. The displacement is the difference between the object's ending point and starting point.

(b) $\int_a^b |v(t)|\,dt$ measures the total distance traveled between $t = a$ and $t = b$.

If $v(t)$ is always positive, *displacement* and *distance traveled* are the same.

Example 1 (Easy area problem): Find the area of the region in the xy-plane bounded above by the graph of the function $f(x) = 2$, below by the x-axis, on the left by the line $x = 1$, and on the right by the line $x = 5$.

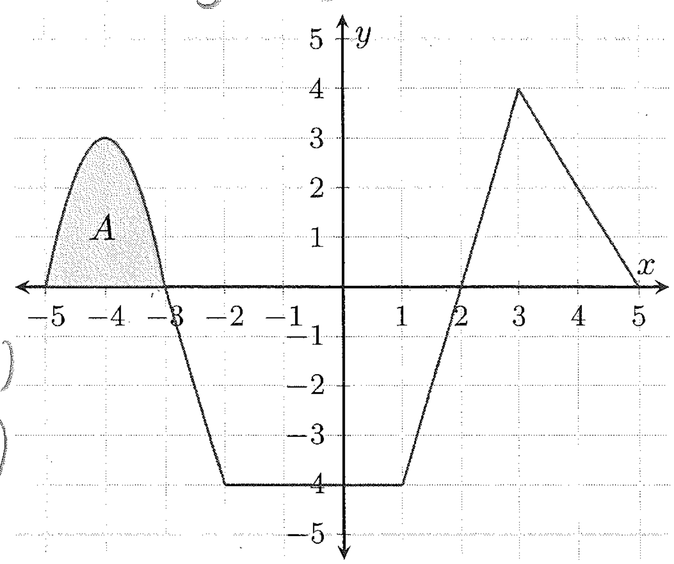


Area = base · height = 4(2) = 8

Example 2 (Easy distance traveled problem): Suppose a car is traveling due east at a constant velocity of 55 miles per hour. How far does the car travel between noon and 2:00 pm?

Distance = rate · time = $55 \frac{\text{mi}}{\text{hr}} \cdot 2 \text{ hr}$ = 110 miles

Example 3: Use the graph of $f(x)$ shown to find the following integrals, given that the shaded region has area A. (Area below x-axis counts as negative)



(a) $\int_2^5 f(x)\,dx$ = area of triangle

$= \frac{1}{2}bh = \frac{1}{2}(3)(4)$

$= \frac{12}{2} = \boxed{6}$

(b) $\int_{-3}^2 f(x)\,dx$ = −(Area of trapezoid)

$= -\left(h \cdot \frac{b_1+b_2}{2}\right) = -\left(4 \cdot \frac{5+3}{2}\right)$

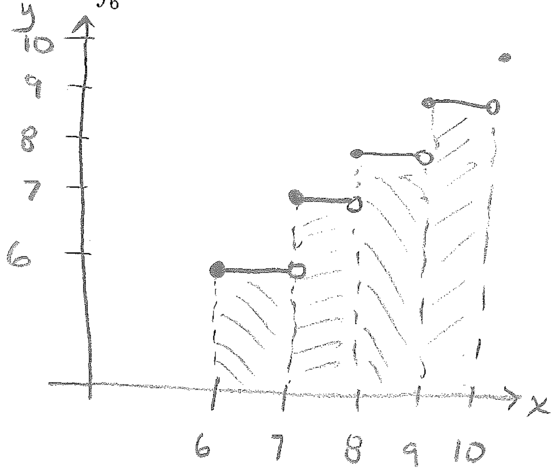
$= -\left(4 \cdot \frac{8}{2}\right) = \boxed{-16}$

(c) $\int_{-3}^5 f(x)\,dx = \underbrace{\int_{-3}^{2} f(x)\,dx}_{\text{trapezoid below x-axis}} + \underbrace{\int_{2}^{5} f(x)\,dx}_{\text{triangle above x-axis}} = -16 + 6 = \boxed{-10}$

(d) $\int_{-5}^5 f(x)\,dx = \underbrace{\int_{-5}^{-3} f(x)\,dx}_{\text{shaded region}} + \underbrace{\int_{-3}^{5} f(x)\,dx}_{\text{answer to (c)}} = \boxed{A - 10}$

Example 4: Suppose $f(x)$ is the greatest integer function, i.e., $f(x)$ equals the greatest integer less than or equal to x. So for example $f(2.3) = 2$, $f(4) = 4$, and $f(6.9) = 6$.

Find $\int_6^{10} f(x)\,dx$.



$$\int_6^{10} f(x)\,dx = \text{area between } x\text{-axis and graph of } f(x)$$

$$= \text{area of 4 rectangles}$$

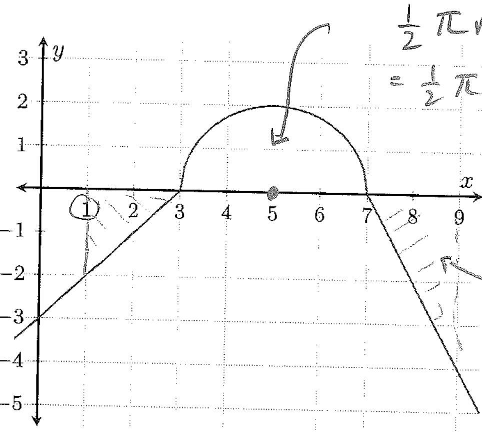
$$= 6(1) + 7(1) + 8(1) + 9(1)$$

$$= 6 + 7 + 8 + 9$$

$$= \boxed{30}$$

Example 5: Consider $g(x)$ shown here. The graph from $x = 3$ to $x = 7$ is a semicircle.

Find $\int_1^9 g(x)\,dx$.



Semicircle area:

$\frac{1}{2}\pi r^2$

$= \frac{1}{2}\pi(2)^2 = \frac{4\pi}{2}$

$= \boxed{2\pi}$

triangle area $= \frac{1}{2}bh$

$= \frac{1}{2}(2)(2)$

$= 2$

triangle area

$= \frac{1}{2}bh$

$= \frac{1}{2}(2)(4)$

$= 4$

$$= \int_1^3 g(x)\,dx + \int_3^7 g(x)\,dx + \int_7^9 g(x)\,dx$$

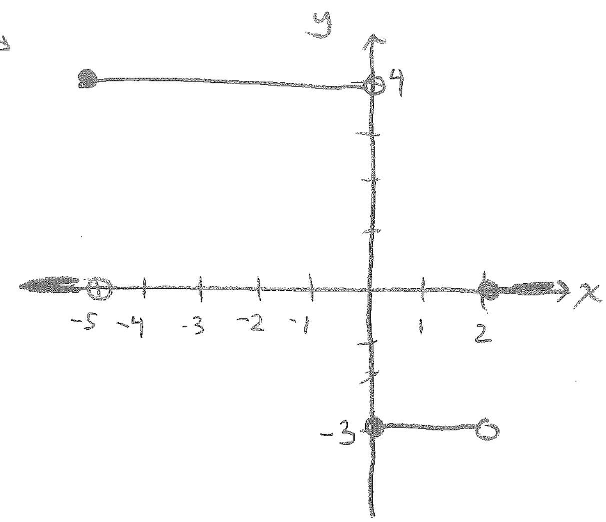
$$= -2 + 2\pi - 4$$

triangle below axis | semicircle above x-axis | triangle below x-axis

$$= \boxed{2\pi - 6}$$

$\boxed{\textbf{Example 6:}}$ Let

$y = f(x) \longrightarrow$



$$f(x) = \begin{cases} 0 & \text{if } x < -5 \\ 4 & \text{if } -5 \le x < 0 \\ -3 & \text{if } 0 \le x < 2 \\ 0 & \text{if } x \ge 2 \end{cases}$$

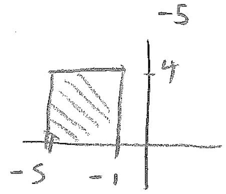
and $g(x) = \displaystyle\int_{-5}^{x} f(t)\, dt$.

Determine the value of each of the following:

(a) $g(-10) = \displaystyle\int_{-5}^{-10} f(t)\,dt = \int_{-5}^{10} 0\, dt = \boxed{0}$

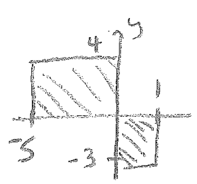
for $-10 \le x \le -5$, $f(x) = 0$ always

(b) $g(-1) = \displaystyle\int_{-5}^{-1} f(t)\,dt = \int_{-5}^{-1} 4\, dt = \underset{\text{base}\,\cdot\,\text{height}}{4(4)} = \boxed{16}$



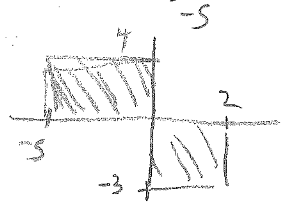
$\leftarrow$ area of rectangle

(c) $g(1) = \displaystyle\int_{-5}^{1} f(t)\,dt = \underbrace{\int_{-5}^{0} f(t)\,dt}_{\text{rectangle above x-axis}} + \underbrace{\int_{0}^{1} f(t)\,dt}_{\text{rectangle below x-axis}}$



$= 5(4) - 1(3) = 20 - 3 = \boxed{17}$

(d) $g(6) = \displaystyle\int_{-5}^{6} f(t)\,dt = \int_{-5}^{0} f(t)\,dt + \int_{0}^{2} f(t)\,dt + \int_{2}^{6} f(t)\,dt$



$= 5(4) - 2(3) + 0$

$= 20 - 6 + 0 = \boxed{14}$

(e) What is the absolute maximum of $g(x)$?

$\hookrightarrow$ Largest possible area under $f(t)$ is $\boxed{20}$

▶ **Some properties of definite integrals:**

1. $\int_a^a f(x)\,dx = 0$

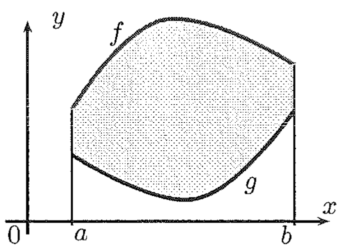
2. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

3. $\int_a^b (f(x) \pm g(x))\,dx = \left(\int_a^b f(x)\,dx\right) \pm \left(\int_a^b g(x)\,dx\right)$

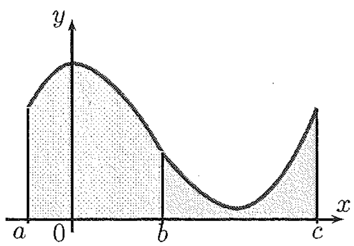
4. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

5. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Geometric illustration of some of the above properties:



Property **3.** says that if f and g are positive-valued functions with f greater than g, then $\int_a^b (f(x) - g(x))\,dx$ gives the area between the graphs of f and g. However, we can rephrase this as the area under g subtracted from the area under f, which is given by $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.



Property **4.** says that if $f(x)$ is a positive valued function, then the area underneath the graph of $f(x)$ between a and b plus the area underneath the graph of $f(x)$ between b and c equals the area underneath the graph of $f(x)$ between a and c.

Property **5.** follows from Properties **4.** and **1.** by letting $c = a$.

Example 7: Using the graph of $f(x)$ from Example 3, find the integral $\int_2^5 5f(x)\,dx$.

$$\int_2^5 5f(x)\,dx = 5\underbrace{\int_2^5 f(x)\,dx}_{\text{answer to Example 3(a) is 6}} = 5(6) = \boxed{30}$$

property 2 above allows us to factor out the coefficient

Example 8: Let

$$\int_1^4 f(x)\,dx = 3, \qquad \int_1^9 f(x)\,dx = -4, \qquad \int_1^4 g(x)\,dx = 2, \qquad \int_1^9 g(x)\,dx = 8, \qquad \int_6^9 g(x)\,dx = 3.$$

Use these values to evaluate the given definite integrals.

(a) $\displaystyle\int_1^4 (f(x) - g(x))\,dx = \int_1^4 f(x)\,dx - \int_1^4 g(x)\,dx$

$= 3 - 2 = \boxed{1}$

(b) $\displaystyle\int_9^1 (f(x) + g(x))\,dx = -\int_1^9 (f(x) + g(x))\,dx$

$\displaystyle = -\int_1^9 f(x)\,dx - \int_1^9 g(x)\,dx = -(-4) - 8 = 4 - 8 = \boxed{-4}$

(c) $\displaystyle\int_4^9 (7f(x) + 10g(x))\,dx = 7\int_4^9 f(x)\,dx + 10\int_4^9 g(x)\,dx$

$= 7(-7) + 10(6) = -49 + 60 = 11$

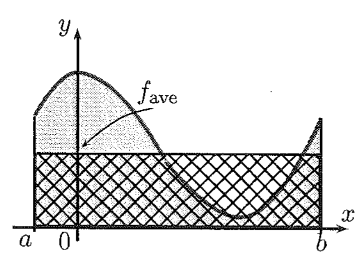
See next page

(d) $\displaystyle\int_4^6 (g(x) - 5)\,dx = \int_4^6 g(x)\,dx - \int_4^6 5\,dx = 3 - 10 = -7$

see next page!

▶ **Average Values:** The average of finitely many numbers $y_1, y_2, \ldots, y_n$ is $y_{\text{ave}} = \dfrac{y_1 + y_2 + \cdots + y_n}{n}$. What if we are dealing with infinitely many values? More generally, how can we compute the average of a function f defined on an interval?

Average of a function: The average of a function f on an interval $[a, b]$ equals the integral of f over the interval divided by the length of the interval:



$$f_{\text{ave}} = \frac{\displaystyle\int_a^b f(x)\,dx}{b - a}.$$

Geometric meaning: If f is a positive valued function, f_{ave} is that number such that the rectangle with base $[a, b]$ and height f_{ave} has the same area as the region underneath the graph of f from a to b.

(8) (c) To find $\int_4^9 f(x)\,dx$:

We know $\int_1^4 f(x)\,dx + \int_4^9 f(x)\,dx = \int_1^9 f(x)\,dx$

$\Rightarrow \quad 3 + \int_4^9 f(x)\,dx = -4$

$\Rightarrow \quad \int_4^9 f(x)\,dx = -4 - 3 = -7$

To find $\int_4^9 g(x)\,dx$:

We know $\int_1^4 g(x)\,dx + \int_4^9 g(x)\,dx = \int_1^9 g(x)\,dx$

$\Rightarrow \quad 2 + \int_4^9 g(x)\,dx = 8$

$\Rightarrow \quad \int_4^9 g(x)\,dx = 8 - 2 = 6$

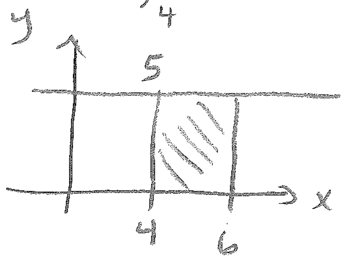
(d) To find $\int_4^6 g(x)\,dx$: We know

found in (c) above

$$\int_4^6 g(x)\,dx + \int_6^9 g(x)\,dx = \int_4^9 g(x)\,dx$$

$\Rightarrow \quad \int_4^6 g(x)\,dx + 3 = 6 \Rightarrow \int_4^6 g(x)\,dx = 6 - 3 = 3$

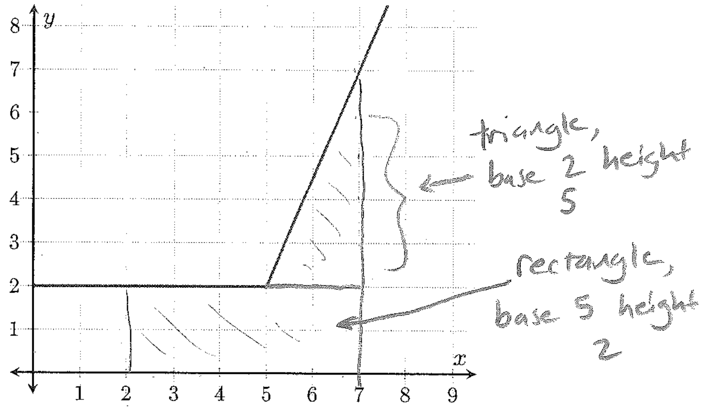
To find $\int_4^6 5\,dx$: graph the function $y = 5$



area of rectangle

$= \text{base} \cdot \text{height}$

$= 2\,(5) = 10$



Example 9: Suppose $f(x) = \begin{cases} 2 & \text{if } x \le 5 \\ \frac{1}{2}(5x-21) & \text{if } x > 5. \end{cases}$

Find the average value of $f(x)$ over the interval $[2,7]$.

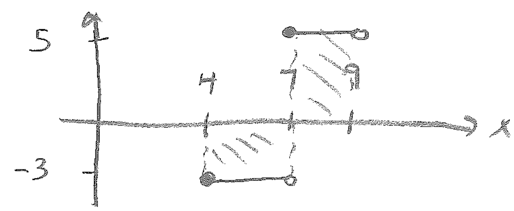
$$\int_2^7 f(x)\,dx = \text{area of rectangle} + \text{area of triangle}$$

$$= 5(2) + \tfrac{1}{2}(2)(5)$$

$$= 10 + 5 = 15.$$

Average value over $[2,7]$

$$= \frac{1}{7-2}\int_2^7 f(x)\,dx = \frac{1}{5}(15) = 3$$



Example 10: Suppose $f(x) = \begin{cases} -3 & \text{if } 4 \le x < 7 \\ 5 & \text{if } 7 \le x < 9. \end{cases}$

should be $\le$

(a) Find the average value of $f(x)$ on the interval $[4,9]$.

$$\int_4^9 f(x)\,dx = -3(3) + 2(5) = -9 + 10 = 1$$

Average value over $[4,9]$

$$= \frac{1}{9-4}\int_4^9 f(x)\,dx = \frac{1}{5}(1) = \frac{1}{5}$$

(b) Find the average rate of change of $f(x)$ on the interval $[4,9]$

$$\text{AROC} = \frac{f(9)-f(4)}{9-4} = \frac{5-(-3)}{5} = \frac{8}{5}$$