Math 123 – Elem. Calculus **Fall 2014** **Name:** SOLUTIONS **Sec.:**______
Exam 4 Bonus Questions

The questions below are bonus questions. You should write your answers on this page. **BOTH THE STEPS YOU SHOW (YOUR WORK) AND YOUR FINAL ANSWER MAY AFFECT YOUR SCORE.** You must show proper, logical, sensible and legible work to be sure you will get full credit. No books or notes may be used. You may use an ACT-approved calculator but NO calculator with a Computer Algebra System (CAS), networking or camera is permitted. Absolutely no cell phone use during the exam is allowed.

You **must turn in this page** with your name even if you do not complete the problems.

1. Suppose we are given the **derivative** of a function: $f'(x)=(x+3)e^{x}$.

   Find interval(s) where $f(x)$ is **concave up**.

Product Rule

$$f''(x) = (x+3)e^{x} + e^{x}(1)$$

(Factor out e^x) $= e^{x}(x+3+1) = e^{x}(x+4)$

e^{x} is never zero, so $f''(x)=0$ when $x=-4$.

	$x<-4$	$x>-4$
f	∩	∪
f''	−	+

test -100 $\quad -4 \quad$ test 100

Final answer: $(-4, \infty)$

2. Find the **average value** of the function $f(x)=x^{3}+3x^{2}+1$ on the interval $[0,2]$.

Average value of $f(x)$ over $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\frac{1}{2-0}\int_0^2 (x^3+3x^2+1)\,dx = \frac{1}{2}\cdot\left(\frac{1}{4}x^4 + x^3 + x\right)\Big|_0^2$$

$$= \frac{1}{2}\cdot\left[\frac{1}{4}\cdot 2^4 + 2^3 + 2 - (0+0+0)\right] = \frac{1}{2}\left[\frac{16}{4} + 8 + 2\right]$$

$$= \frac{1}{2}\left[4+10\right] = \frac{14}{2}$$

Final answer: 7

MA123 — Elem. Calculus	Fall 2014	Name: SOLUTIONS	Sec.: ______
Exam 4	2014-12-16		

Do not remove this answer page — you will turn in the entire exam. You have two hours to do this exam. No books or notes may be used. You may use an ACT-approved calculator during the exam, but NO calculator with a Computer Algebra System (CAS), networking, or camera is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of multiple choice questions. Record your answers on this page. For each multiple choice question, you will need to fill in the box corresponding to the correct answer. For example, if (a) is correct, you must write

(a) (b) (c) (d) (e)

Do not circle answers on this page, but please circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on both this page and in the body of the exam.

GOOD LUCK!

1.	(a)	(b)	**(c)**	(d)	(e)	11.	(a)	(b)	(c)	(d)	**(e)**
2.	(a)	(b)	(c)	(d)	**(e)**	12.	(a)	**(b)**	(c)	(d)	(e)
3.	**(a)**	(b)	(c)	(d)	(e)	13.	**(a)**	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	**(d)**	(e)	14.	**(a)**	(b)	(c)	(d)	(e)
5.	(a)	(b)	(c)	**(d)**	(e)	15.	(a)	(b)	**(c)**	(d)	(e)
6.	(a)	(b)	**(c)**	(d)	(e)	16.	(a)	(b)	**(c)**	(d)	(e)
7.	(a)	(b)	(c)	(d)	**(e)**	17.	(a)	(b)	(c)	**(d)**	(e)
8.	(a)	**(b)**	(c)	(d)	(e)	18.	**(a)**	(b)	(c)	(d)	(e)
9.	(a)	**(b)**	(c)	(d)	(e)	19.	(a)	(b)	(c)	**(d)**	(e)
10.	(a)	**(b)**	(c)	(d)	(e)	20.	(a)	(b)	(c)	(d)	**(e)**

For grading use:

Number Correct	
	(out of 20 problems)

Total	
	(out of 100 points)

Multiple Choice Questions

Show all your work on the page where the question appears. Clearly mark your answer both on the cover page on this exam and in the corresponding questions that follow.

1. Find the limit as n tends to infinity. Here C is a fixed real number.

$$\lim_{n\to\infty} \frac{(3n+1)^2}{7n^2+4n+C}$$

"FOIL" numerator

$$= \lim_{n\to\infty} \frac{9n^2+6n+1}{7n^2+4n+C}$$

since $n \to \infty$, and expression is a single fraction, only terms with the highest power in numerator + denominator matter

$$= \lim_{n\to\infty} \frac{9n^2}{7n^2} = \lim_{n\to\infty} \frac{9}{7} = \frac{9}{7}$$

Possibilities:

(a) $\frac{3}{7} + C$

(b) 0

(c) $\frac{9}{7}$

(d) ∞

(e) $\frac{3}{11+C}$

2. Evaluate the limit as n tends to infinity. Note: you will have to use some of the summation formulas (see formula sheet on backpage) to simplify.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{9k^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{9}{n^2} \sum_{k=1}^{n} k^2$$

cancel one n from top/bottom

$$= \lim_{n\to\infty} \frac{9}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \lim_{n\to\infty} \frac{9(n+1)(2n+1)}{6n^2}$$

$$= \lim_{n\to\infty} \frac{9(2n^2+3n+1)}{6n^2} = \lim_{n\to\infty} \frac{18n^2+27n+9}{6n^2} = \lim_{n\to\infty} \frac{18n^2}{6n^2}$$

$$= 18/6 = 3$$

Possibilities:

(a) 1

(b) 2

(c) 5

(d) 4

(e) 3

3. The integral

$$\int_4^{10} x^2 \, dx$$

width

$= \Delta x = \frac{b-a}{n} = \frac{10-4}{n} = \frac{6}{n}$

is computed as the limit of the sum

$$\sum_{k=1}^{n} \underbrace{\frac{A}{n}}_{\text{width}} \underbrace{\left(4 + k\frac{A}{n}\right)^2}_{\text{height}}$$

x-values for right endpoints of k^{th} rectangle:

$x_k = a + k\Delta x = 4 + k \cdot \frac{6}{n}$

height:

$(x_k)^2 = \left(4 + k\frac{6}{n}\right)^2$

What value should be used for A?

Possibilities:

(a) 6
(b) 4
(c) 10
(d) 14
(e) 312

---

4. Assuming $x > 0$, evaluate the definite integral

$$\int_2^x \frac{4}{t} \, dt = 4\ln|t| \Big|_2^x$$

$= 4\ln|x| - 4\ln|2|$

(Since x and 2 are both > 0, absolute values are not required in this case, but they're not wrong.)

Possibilities:

(a) $-\frac{4}{x^2} + 1$
(b) $8\sqrt{x} - 8\sqrt{2}$
(c) $4\sqrt{x}$
(d) $4\ln(|x|) - 4\ln(2)$
(e) $\frac{4}{\frac{1}{2}x^2} - 2$

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5. Use the Fundamental Theorem of Calculus to compute the derivative, $F'(x)$, of $F(x)$, if

$$F(x) = \int_1^x (t^4 + t^3 + t^2 + 3t + 8) \, dt$$

replace t's with x's

$F'(x) = x^4 + x^3 + x^2 + 3x + 8$

Possibilities:

(a) $\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{3}{2}x + 8x$
(b) $x^4 + x^3 + x^2 + 3x$
(c) $4x^3 + 3x^2 + 2x + 3$
(d) $x^4 + x^3 + x^2 + 3x + 8$
(e) $4x^3 + 3x^2 + 2x + 11$

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6. Find the value of x at which

$$F(x) = \int_6^x \left(-t^4 - t^2 - 2\right)\, dt$$

takes its minimum value on the interval $[8, 500]$.

Possibilities:

(a) $\frac{8196}{5}$

(b) 6

(c) 500

(d) 8

(e) 1334

$$F'(x) = -x^4 - x^2 - 2 = -(\underbrace{x^4 + x^2 + 2}_{\text{always positive}})$$

so $F'(x)$ is always negative.

Since $F'(x) < 0$ always, $F(x)$ is always decreasing.

Thus, its minimum value on $[8, 500]$ will occur at $x = 500$.

7. Evaluate the integral

$$\int_0^x (t+4)^3\, dt$$

$u = t + 4$

$\frac{du}{dt} = 1 \qquad dt = du$

Possibilities:

(a) $\frac{1}{4}x^4 - \frac{4^4}{4}$

(b) $\frac{1}{4}x^4$

(c) $\frac{1}{3}(x+4)^3 - \frac{4^3}{3}$

(d) $4(x+4)^4 - 3 \cdot 4^4$

(e) $\frac{1}{4}(x+4)^4 - \frac{4^4}{4}$

$\left[\begin{array}{l} \text{when } t = 0, \quad u = 0 + 4 = 4 \\ \text{when } t = x, \quad u = x + 4 = x + 4 \end{array}\right]$

$$\int_4^{x+4} u^3\, du = \frac{1}{4}u^4 \Big|_4^{x+4}$$

$$= \frac{1}{4}(x+4)^4 - \frac{1}{4} \cdot 4^4$$

8. A train travels along a track and its speed (in miles per hour) is given by $s(t) = 48t$ for the first half hour of travel. Its speed is constant and equal to $s(t) = 24$ after the first half hour. (Here time t is measured in hours.) How far (in miles) does the train travel in the first hour of travel?

Possibilities:

(a) 6 miles

(b) 18 miles

(c) 24 miles

(d) 12 miles

(e) 48 miles

During the 1st half-hour, distance $= \int_0^{\frac{1}{2}} 48t\, dt$

$$= 48 \cdot \frac{t^2}{2} \Big|_0^{\frac{1}{2}} = 24t^2 \Big|_0^{\frac{1}{2}} = 24\left(\tfrac{1}{2}\right)^2 - 0 = \frac{24}{4} = 6 \text{ miles}$$

During the 2nd half-hour, distance = rate · time $= 24 \cdot \frac{1}{2} = 12$ miles.

Total distance traveled = $6 + 12 = 18$ miles.

9. Evaluate the indefinite integral

distribute

$$\int t^3(t+16)\ dt = \int (t^4 + 16t^3)\, dt$$

$$= \frac{1}{5}t^5 + 16 \cdot \frac{1}{4}t^4 + C$$

$$= \frac{1}{5}t^5 + 4t^4 + C$$

Possibilities:

(a) $\frac{1}{4}t^4 + \dfrac{16}{3}t^3 + C$, for any number C

(b) $\frac{1}{5}t^5 + 4t^4 + C$, for any number C

(c) $5t^5 + 64t^4 + C$, for any number C

(d) $\frac{1}{4}t^4 + \frac{1}{2}t^2 + C$, for any number C

(e) $\left(\frac{1}{4}t^4\right)\left(\frac{1}{2}t^2 + 16t\right) + C$, for any number C

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10. Find the average rate of change of $f(x) = \sqrt{x}$ from $x = 9$ to $x = 49$.

$$\text{AROC} = \frac{f(49) - f(9)}{49 - 9} = \frac{\sqrt{49} - \sqrt{9}}{40}$$

Stop here

$$= \cancel{\frac{7-3}{40} = \frac{4}{40} = \frac{1}{10}}$$

Possibilities:

(a) $\dfrac{\sqrt{49} - \sqrt{9}}{\sqrt{9} - \sqrt{49}}$

(b) $\dfrac{\sqrt{49} - \sqrt{9}}{49 - 9}$

(c) $\dfrac{\log(9) + \log(49)}{2}$

(d) $\dfrac{1}{9} - \dfrac{1}{49}$

(e) $\dfrac{1}{2}(49)^{-1/2} - \dfrac{1}{2}(9)^{-1/2}$

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11. Compute $\displaystyle\lim_{t \to 8} \frac{t^2 - 6t - 16}{t^2 - 3t - 40}$

test $t = 8$: $\dfrac{8^2 - 6(8) - 16}{8^2 - 3(8) - 40} = \dfrac{0}{0}$, must do more work!

$$= \lim_{t \to 8} \frac{(t-8)(t+2)}{(t-8)(t+5)} = \lim_{t \to 8} \frac{t+2}{t+5}$$

We know in advance that $t-8$ is a factor of each

$$= \frac{8+2}{8+5} = \frac{10}{13}$$

Possibilities:

(a) $\frac{6}{13}$

(b) $\frac{7}{13}$

(c) $\frac{8}{13}$

(d) The limit does not exist.

(e) $\frac{10}{13}$

12. A train travels from city A to city B, then travels from city B to city C. The train leaves city A at time 1:00pm and arrives at city B at 4:00pm. The train leaves city B at 5:00pm and arrives at city C at 7:00pm. The average velocity of the train, while travelling from A to B, was 30 miles per hour. The average velocity of the train, while travelling from B to C, was 50 miles per hour. What was the average velocity of the train from city A to city C, including the wait at city B?

Possibilities:

(a) 80 miles per hour

(b) (95/3) miles per hour

(c) (98/3) miles per hour

(d) 10 miles per hour

(e) 40 miles per hour

Distance from A to B = rate · time = 30 mph (3 hrs) = 90

Distance from B to C = 50 mph · 2 hrs = 100 miles.

Avg. velocity from A to C = $\frac{\text{total distance}}{\text{total time}}$

$= \frac{190 \text{ miles}}{6 \text{ hrs}} = \frac{190}{6} = \frac{95}{3}$ miles per hour

13. The tangent line to the graph of f at $x = 6$ has equation $y = 7(x - 6) + 9$. Find $f(6)$ and $f'(6)$.

Possibilities:

(a) $f(6) = 9, \quad f'(6) = 7$

(b) $f(9) = 7, \quad f'(9) = 6$

(c) $f(6) = 7, \quad f'(6) = 9$

(d) $f(7) = 9, \quad f'(7) = 6$

(e) $f(9) = 6, \quad f'(9) = 7$

At $x = 6$, the tangent line has the same slope and y-value as $f(x)$.

Thus, $y = 7(6-6)+9 = 0+9 = f(6)$,

and the slope $m = 7$ is $f'(6)$.

14. The graph of $y = f(x)$ is shown below. The function is differentiable, except at $x =$

Possibilities:

(a) x=1 and x=4

(b) x=4 only

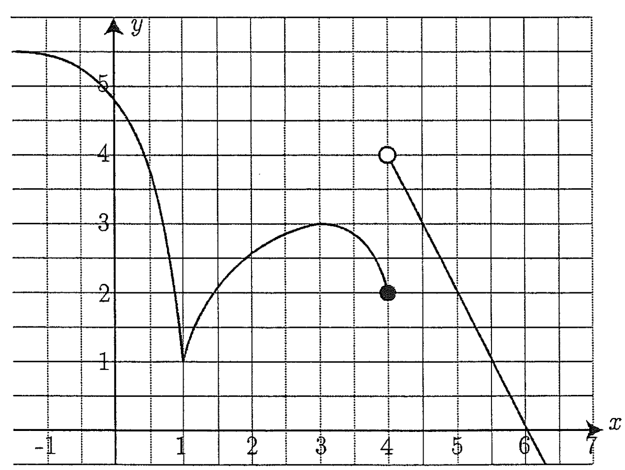
(c) x=1 only

(d) x=1, x=3, and x=4

(e) x=1, x=3, x=4, and x=6

at $x = 1$, graph has a "sharp point"

at $x = 4$, graph is not continuous.



15. Find the derivative, $f'(x)$, of $f(x) = 4x^3$

$f'(x) = 12x^2$

Possibilities:

(a) $-3x^{(1/4)}$

(b) $\frac{1}{4}x^4$

(c) $12x^2$

(d) x^4

(e) x^3

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16. Suppose that $f(x) = (13x - g(x))^3$, but that the formula for $g(x)$ is too complicated to write down. When $x = 2$, the value and derivative of g are measured: $g(2) = 7$, and $g'(2) = 5$. What is $f'(2)$?

Possibilities:

(a) 1152

(b) 1083

(c) 8664

(d) 147

(e) 192

$$f'(x) = \overbrace{3(13x - g(x))^2}^{\text{power rule}} \cdot \overbrace{(13 - g'(x))}^{\text{inside}}$$

$$f'(2) = 3(13 \cdot 2 - g(2))^2 \cdot (13 - g'(2))$$ replace x with 2

$$= 3(26 - 7)^2 \cdot (13 - 5)$$

$$= 3 \cdot 19^2 \cdot 8 = 8664$$

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17. Find the derivative, $f'(x)$, if $f(x) = \sqrt{3x + x^8}$. $= (3x + x^8)^{\frac{1}{2}}$

Possibilities:

(a) $\frac{1}{2}(3 + 8x^7)^{(-1/2)}$

(b) $-\frac{1}{2}(3x + x^8)^{(1/2)}(3 + 7x^8)$

(c) $\frac{1}{2}(3x + x^8)^{(-1/2)}$

(d) $\frac{1}{2}(3x + x^8)^{(-1/2)}(3 + 8x^7)$

(e) $\frac{1}{2}(3x + x^8)^{(-1/2)}(3 + 8x^7)(8 \cdot 7x^6)$

$$f'(x) = \underbrace{\frac{1}{2}(3x + x^8)^{-\frac{1}{2}}}_{\text{power rule}} \cdot \underbrace{(3 + 8x^7)}_{\text{inside}}$$

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18. Suppose the derivative of $g(t)$ is $g'(t) = 7(t-8)(t-6)(t-9)$. For t in which interval(s) is g increasing?

$g'(t) = 0$ for $t = 6, 8, 9$.

Possibilities:

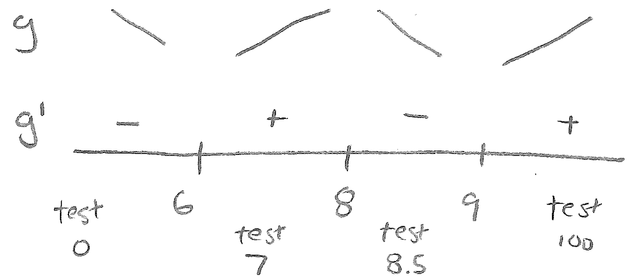
(a) $(6,8) \cup (9,\infty)$

(b) $(-\infty,6) \cup (8,9)$

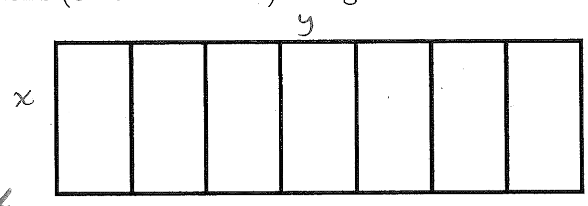
(c) $(-\infty, \frac{23}{3} - \frac{1}{3}\sqrt{7}) \cup (\frac{23}{3} + \frac{1}{3}\sqrt{7}, \infty)$

(d) $(7,6) \cup (8,9)$

(e) $(\frac{23}{3} - \frac{1}{3}\sqrt{7}, \frac{23}{3} + \frac{1}{3}\sqrt{7})$



19. A farmer builds a rectangular pen with 7 vertical partitions (8 vertical sides) using 500 feet of fencing. What is the maximum possible total area of the pen?



Possibilities:

(a) $\frac{31250}{9}$

(b) 6250

(c) 15625

(d) $\frac{15625}{4}$

(e) 500

$A = xy$

$8x + 2y = 500$

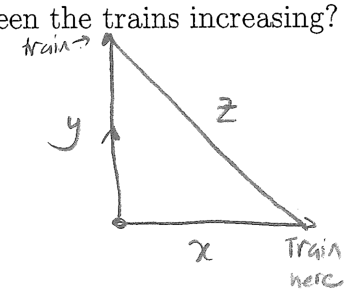
$4x + y = 250 \qquad y = 250 - 4x$

$A = x(250 - 4x) = 250x - 4x^2$ domain: $[0, \frac{125}{2}]$

$A' = 250 - 8x \qquad A' = 0$ if $8x = 250 \Rightarrow x = \frac{250}{8} = \frac{125}{4}$ ft

$A = \frac{125}{4}(250 - 125) = \frac{125 \cdot 125}{4} = \frac{15625}{4}$ ft^2

20. Two trains leave the same station at different times, one traveling due East, and the other traveling due North. At 2pm the eastbound train is traveling at 65 mph and is 400 miles from the station, while the northbound train is traveling at 50 mph and is 300 miles from the station. At what rate is the distance between the trains increasing?



Possibilities:

(a) 82000 mph

(b) $5\sqrt{269}$ mph

(c) 115 mph

(d) 500 mph

(e) 82 mph

$x^2 + y^2 = z^2$

$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$

$2(400) \cdot 65 + 2(300)(50) = 2(500) \frac{dz}{dt}$

x = distance from Eastbound train to station

y = " " Northbound train to station

If $x = 400$ and $y = 300$, then $z = \sqrt{300^2 + 400^2} = 500$

$1000 \frac{dz}{dt} = 82000$

$\frac{dz}{dt} = \frac{82000}{1000} = 82$ mph

Some Formulas

1. **Summation formulas:**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. **Areas:**

   (a) Triangle $A = \frac{bh}{2}$

   (b) Circle $A = \pi r^2$

   (c) Rectangle $A = lw$

   (d) Trapezoid $A = \frac{h_1 + h_2}{2} b$

3. **Volumes:**

   (a) Rectangular Solid $V = lwh$

   (b) Sphere $V = \frac{4}{3}\pi r^3$

   (c) Cylinder $V = \pi r^2 h$

   (d) Cone $V = \frac{1}{3}\pi r^2 h$

4. **Distance:**

   (a) Distance between (x_1, y_1) and (x_2, y_2)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$