MA123 — Elem. Calculus Spring 2015 Final Exam 2015-05-06

Name: SOLUTIONS Sec.: ________

Do not remove this answer page — you will turn in the entire exam. You have two hours to do this exam. No books or notes may be used. You may use an ACT-approved calculator during the exam, but NO calculator with a Computer Algebra System (CAS), networking, or camera is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of two short answer questions and eighteen multiple choice questions. Answer the short answer questions on the back of this page, and record your answers to the multiple choice questions on this page. For each multiple choice question, you will need to fill in the circle corresponding to the correct answer. For example, if (a) is correct, you must write

(a) (b) (c) (d) (e)

Do not circle answers on this page, but please circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on both this page and in the body of the exam.

GOOD LUCK!

3. (a) (b) (c) (d) (e)
4. (a) (b) (c) (d) (e)
5. (a) (b) (c) (d) (e)
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17. (a) (b) (c) (d) (e)
18. (a) (b) (c) (d) (e)
19. (a) (b) (c) (d) (e)
20. (a) (b) (c) (d) (e)

For grading use:

Multiple Choice		Short Answer
(number right)	(5 points each)	(out of 10 points)

Total	
	(out of 100 points)

Spring 2015 Exam 4 Short Answer Questions

Write your answers on this page. You must show clear, appropriate legible work to be sure you will get full credit.

1. Compute the following integral: $\int_1^T \frac{6x^5+x}{x^3}\,dx$.

$$= \int_1^T \left(\frac{6x^5}{x^3} + \frac{x}{x^3}\right) dx$$

$$= \int_1^T (6x^2 + x^{-2})\,dx = 6\cdot\frac{1}{3}x^3 + \left(\frac{1}{-1}\right)x^{-1}\Big|_1^T$$

$$= 2x^3 - \frac{1}{x}\Big|_1^T$$

$$= 2T^3 - \frac{1}{T} - \left(2\cdot 1^3 - \frac{1}{1}\right)$$

Final answer: $2T^3 - \frac{1}{T} - 1$

2. Let $F(x)=\ln(3x+g(x))$. Find the **slope** of the tangent line to the graph of $y=F(x)$ at $x=2$, given $g(2)=11$ and $g'(2)=-1$.

$$F'(x) = \underbrace{\frac{1}{3x+g(x)}}_{\ln\text{ rule}} \cdot \underbrace{(3+g'(x))}_{\text{deriv. of inside}} = \frac{3+g'(x)}{3x+g(x)}$$

$$m = F'(2) = \frac{3+g'(2)}{3(2)+g(2)} = \frac{3-1}{6+11} = \frac{2}{17}$$

Final answer: $2/17$

Name: SOLUTIONS

Multiple Choice Questions

Show all your work on the page where the question appears. Clearly mark your answer both on the cover page on this exam and in the corresponding questions that follow.

3. Find the limit as n tends to infinity. Here C is a fixed real number.

$$\lim_{n\to\infty} \frac{(3n+1)^2}{Cn^2+7n+4}$$

$$= \lim_{n\to\infty} \frac{9n^2+6n+1}{Cn^2+7n+4}$$ "FOIL"

$$= \lim_{n\to\infty} \frac{9n^2}{Cn^2}$$ for $n\to\infty$, only terms with the highest power matter

$$= \lim_{n\to\infty} \frac{9}{C}$$ cancel the n^2

$$= \frac{9}{C}$$ Limit of a constant

Possibilities:

(a) 0

(b) ∞

(c) $\frac{9}{C^2}$

(d) $\frac{3}{C+11}$

(e) $\frac{9}{C}$

4. Evaluate the limit as n tends to infinity. Note: you will have to use some of the summation formulas (see formula sheet on backpage) to simplify.

$\boxed{\text{Option 1: Apply formulas}}$

pull this outside — split in two

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(8 + k\frac{6}{n}\right)\frac{6}{n} = \lim_{n\to\infty} \frac{6}{n}\left(\sum_{k=1}^{n} 8 + \sum_{k=1}^{n} k\frac{6}{n}\right)$$

$$= \lim_{n\to\infty} \frac{6}{n}\left(8n + \frac{6}{n}\sum_{k=1}^{n} k\right) = \lim_{n\to\infty} \frac{6}{n}\left(8n + \frac{6}{n}\cdot\frac{n(n+1)}{2}\right)$$

formula for constant — pull outside — formula

$$= \lim_{n\to\infty} \frac{6}{n}\left(8n + \underbrace{3(n+1)}_{\text{simplify}}\right) = \lim_{n\to\infty} \frac{6}{n}(11n+3)$$

$$= \lim_{n\to\infty} \frac{66n+18}{n} = \lim_{n\to\infty} \frac{66n}{n} = 66$$

$\boxed{\text{OPTION 2}}$: Recognize this limit as $\int_8^{14} x\,dx = \frac{1}{2}x^2\Big|_8^{14} = \frac{1}{2}(14)^2 - \frac{1}{2}(8)^2 = 66$

Possibilities:

(a) 0

(b) 66

(c) 8

(d) 6

(e) 84

5. Assuming $x > 0$, evaluate the definite integral

$$\int_7^x \frac{4}{t}\ dt$$

$= 4\ln|t| \Big|_7^x$

$= 4\ln|x| - 4\ln 7$

Possibilities:

(a) $-\frac{4}{x^2} + \frac{4}{49}$

(b) $4\sqrt{x}$

(c) $4\ln(|x|) - 4\ln(7)$

(d) $8\sqrt{x} - 8\sqrt{7}$

(e) $\frac{4}{\frac{1}{2}x^2} - \frac{8}{49}$

6. Use the Fundamental Theorem of Calculus to compute the derivative, $F'(x)$, of $F(x)$, if

$$F(x) = \int_1^{\boxed{x+9}} \left(t^2 + 8t + 2\right)\ dt$$

$F'(x) = \left[(x+9)^2 + 8(x+9) + 2\right] \cdot 1$

replace each t with the upper limit of integration

deriv. of x+9 from chain rule

Possibilities:

(a) $\frac{1}{3}x^3 + \frac{8}{2}x^2 + 2x - \left(\frac{1}{3}1^3 + \frac{8}{2}1^2 + 2(1)\right)$

(b) $(x+9)^2 + 8(x+9) + 2$

(c) $2x + 8$

(d) $\frac{1}{3}(x+9)^3 + \frac{8}{2}(x+9)^2 + 2(x+9) - \left(\frac{1}{3}1^3 + \frac{8}{2}1^2 + 2(1)\right)$

(e) $x^2 + 8x + 2$

7. Find the value of x at which

$$F(x) = \int_2^x \left(-t^4 - t^2 - 8\right)\ dt$$

takes its minimum value on the interval $[9, 300]$.

Possibilities:

(a) 28

(b) $\frac{376}{15}$

(c) 2

(d) 300

(e) 9

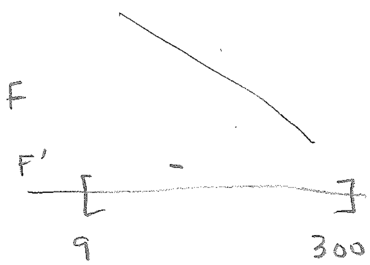
Using the fundamental Theorem,

$F'(x) = -x^4 - x^2 - 8 = -(x^4 + x^2 + 8)$

Notice $F'(x)$ is always negative.

Thus $F(x)$ is always decreasing.

$F(x)$ will take its minimum value at $x = 300$.



8. Evaluate the integral

$$\int_0^x (8t+9)^2\ dt$$

Possibilities:

(a) $\frac{1}{3}x^3 - \frac{9^3}{3}$

(b) $\frac{1}{3}(8x+9)^3 - \frac{9^3}{3}$

(c) $\frac{1}{2}(8x+9)^2 - \frac{9^2}{2}$

(d) $3(8x+9)^3 - 2 \cdot 9^3$

(e) $\frac{1}{8(3)}(8x+9)^3 - \frac{9^3}{8(3)}$

$u = 8t + 9$

$\frac{du}{dt} = 8 \Rightarrow dt = \frac{1}{8} du$

If $t = 0$, $u = 8(0) + 9 = 9$.

If $t = x$, $u = 8x + 9$.

$$\int_9^{8x+9} u^2 \cdot \frac{1}{8} du = \frac{1}{8} \cdot \frac{1}{3} u^3 \Big|_9^{8x+9}$$

$$= \frac{1}{8} \cdot \frac{1}{3} u^3 \Big|_9^{8x+9} = \underline{\frac{1}{8} \cdot \frac{1}{3}(8x+9)^3 - \frac{1}{8} \cdot \frac{1}{3} \cdot 9^3}$$

9. Suppose a rock is dropped from a martian cliff. After t seconds, its speed in feet per second $v(t) = \frac{61}{5}t$, at least until it lands. If the rock lands after 10 seconds, how high (in feet) is the cliff?

Possibilities:

(a) 5 feet

(b) 610 feet

(c) 122 feet

(d) 10 feet

(e) $\frac{61}{50}$ feet

Height of the cliff is the distance traveled in 10 seconds. Distance traveled is the area under the velocity curve.

$$\int_0^{10} v(t)\,dt = \int_0^{10} \frac{61}{5} t\,dt = \frac{61}{5}\cdot\frac{1}{2}t^2 \Big|_0^{10}$$

$$= \frac{61}{10} t^2 \Big|_0^{10} = \frac{61}{10}\cdot 10^2 - \frac{61}{10}\cdot 0^2$$

$$= 61(10) = 610 \text{ feet}$$

10. Compute $\lim_{t\to 2} \dfrac{t^2 - 9t + 14}{t^2 - 10t + 16}$

Possibilities:

(a) $\frac{5}{6}$

(b) 1

(c) $\frac{7}{6}$

(d) $\frac{4}{3}$

(e) The limit does not exist.

test $t = 2$: $\dfrac{4 - 18 + 14}{4 - 20 + 16} = \dfrac{0}{0}$: do more work!

$$\lim_{t\to 2} \frac{(t-2)(t-7)}{(t-2)(t-8)}$$

$$= \lim_{t\to 2} \frac{t-7}{t-8}$$ test again

$$= \frac{2-7}{2-8} = \frac{-5}{-6} = \frac{5}{6}$$

11. Find the average rate of change of $f(x) = \ln x$ from $x = 64$ to $x = 81$.

Possibilities:

(a) $\dfrac{\ln(64) + \ln(81)}{2}$

(b) $\frac{1}{2}(81)^{-1} - \frac{1}{2}(64)^{-1}$

(c) $\dfrac{1}{64} - \dfrac{1}{81}$

(d) $\dfrac{\ln(81) - \ln(64)}{81 - 64}$

(e) $\dfrac{\ln(81) - \ln(64)}{\ln(64) - \ln(81)}$

$$\text{AROC} = \frac{f(b) - f(a)}{b - a} = \frac{f(81) - f(64)}{81 - 64} = \frac{\ln 81 - \ln 64}{81 - 64}$$

12. Solve the equation $5x^2 + 106xy + 3y = 2$ for y in terms of x

Possibilities:

(a) $y = \dfrac{-106 \pm \sqrt{11176}}{10}$

(b) $y = \dfrac{106x + 3}{5x^2 - 2}$

(c) $y = \dfrac{2 - 5x^2}{106x + 3}$

(d) $y = \dfrac{2 - 5x^2 - 106x}{3}$

(e) $y = \dfrac{5x^2 - 2}{106x + 3}$

$$5x^2 + 106xy + 3y = 2$$
$$106xy + 3y = 2 - 5x^2$$
$$y(106x + 3) = 2 - 5x^2$$
$$y = \frac{2 - 5x^2}{106x + 3}$$

13. The tangent line to the graph of f at $x = 8$ has equation $y = 9(x - 8) + 2$. Find $f(8)$ and $f'(8)$.

Possibilities:

(a) $f(9) = 2$, $f'(9) = 8$

(b) $f(8) = 9$, $f'(8) = 2$

(c) $f(2) = 8$, $f'(2) = 9$

(d) $f(8) = 2$, $f'(8) = 9$

(e) $f(2) = 9$, $f'(2) = 8$

$y = 9x - 72 + 2$

$y = 9x - 70$

At $x = 8$, the function $f(x)$ and the tangent line have the same y-value, so $y = \underline{f(8)} = 9(8) - 70 = 72 - 70 = \underline{2}$.

The slope of the tangent line at $x = 8$ is $f'(8)$. Thus $\underline{f'(8) = 9}$.

14. The graph of $y = f(x)$ is shown below. The function is continuous, except at $x =$

Possibilities:

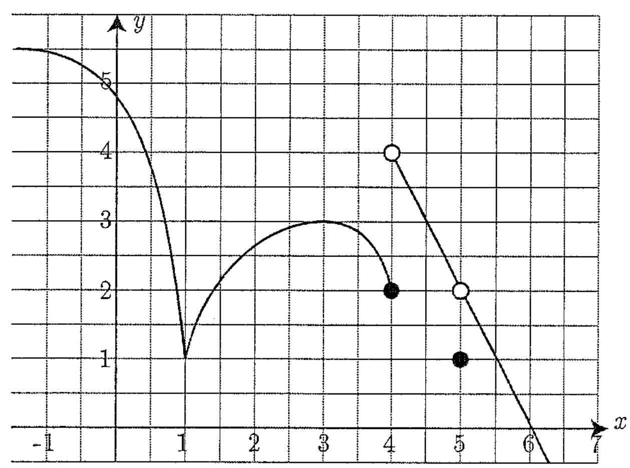
(a) x=4 and x=5

(b) x=1 and x=3

(c) x=1, x=4, and x=5

(d) x=1, x=3, x=4, and x=5

(e) x=4 only



Discontinuities:

At $x = 4$, $\lim_{x \to 4} f(x)$ DNE. ("jump")

At $x = 5$, $\lim_{x \to 5} f(x) \neq f(5)$. ("hole")

(Note: At $x = 1$, the graph has a "sharp point." This is a point where $f(x)$ fails to be differentiable, but $f(x)$ is continuous at $x = 1$.)

15. Find the derivative, $f'(x)$, of $f(x) = 4x^3$

Possibilities:

$f'(x) = 4 \cdot 3x^2 = 12x^2$

(a) $-3x^{(1/4)}$

(b) x^3

(c) $12x^2$

(d) x^4

(e) $\frac{1}{4}x^4$

16. Suppose that $f(x) = \cancel{\log(g(x))}$ $\ln(g(x))$, but that the formula for $g(x)$ is too complicated to write down. When $x = 2$, the value and derivative of g are measured: $g(2) = 3$, and $g'(2) = 9$. What is $f'(2)$?

Possibilities:

$f(x) = \ln(g(x))$

$f'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$

$f'(2) = \frac{g'(2)}{g(2)} = \frac{9}{3} = 3$

(a) $\frac{2}{9}$

(b) 3

(c) $\frac{1}{2}$

(d) $\frac{9}{2}$

(e) $\frac{1}{3}$

17. Find the derivative, $f'(x)$, if $f(x) = \sqrt{9x + x^3}$.

$$f(x) = (9x + x^3)^{\frac{1}{2}}$$

rewrite the radical as a fractional exp.

Possibilities:

(a) $\frac{1}{2}(9x + x^3)^{(-1/2)}(9 + 3x^2)$

(b) $-\frac{1}{2}(9x + x^3)^{(1/2)}(9 + 2x^3)$

(c) $\frac{1}{2}(9x + x^3)^{(-1/2)}$

(d) $\frac{1}{2}(9 + 3x^2)^{(-1/2)}$

(e) $\frac{1}{2}(9x + x^3)^{(-1/2)}(9 + 3x^2)(3 \cdot 2x^1)$

$$f'(x) = \underbrace{\frac{1}{2}(9x + x^3)^{-\frac{1}{2}}}_{\text{power rule}} \cdot \underbrace{(9 + 3x^2)}_{\text{deriv. of inside}}$$

18. Suppose the derivative of $g(t)$ is $g'(t) = -9(t - 8)(t - 2)(t - 3)$. For t in which interval(s) is g increasing?

Possibilities:

$g'(t) = 0$ for $t = 2, 3, 8$

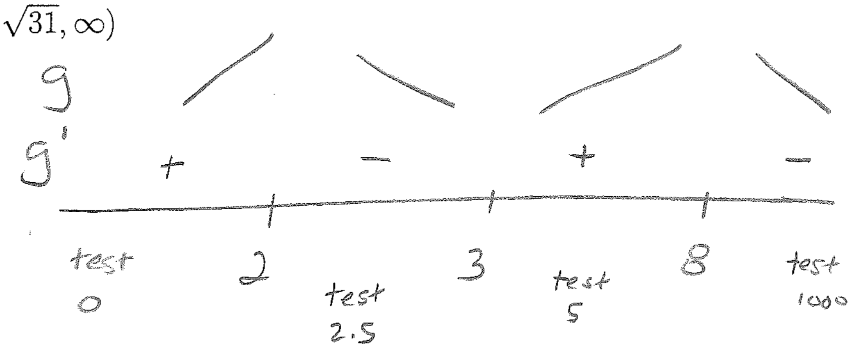
(a) $(-9, 2) \cup (3, 8)$

(b) $(\frac{13}{3} - \frac{1}{3}\sqrt{31}, \frac{13}{3} + \frac{1}{3}\sqrt{31})$

(c) $(-\infty, \frac{13}{3} - \frac{1}{3}\sqrt{31}) \cup (\frac{13}{3} + \frac{1}{3}\sqrt{31}, \infty)$

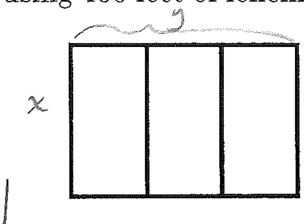
(d) $(-\infty, 2) \cup (3, 8)$

(e) $(2, 3) \cup (8, \infty)$



Test values in the first derivative, caring only about the sign.

19. A farmer builds a rectangular pen with 3 vertical partitions (4 vertical sides) using 400 feet of fencing. What is the maximum possible total area of the pen?



Possibilities:

(a) 4000

(b) 10000

(c) 5000

(d) 400

(e) 20000

$A = xy$

$4x + 2y = 400$

$2x + y = 200$

$y = 200 - 2x$

$A = x(200-2x) = 200x - 2x^2$

Find the critical # :

$A' = 200 - 4x$

$A' = 0$ when $4x = 200$

$\Rightarrow x = 50.$

If $x = 50$, $y = 200 - 2(50) = 100$,
so the area is $50(100) = 5000 \text{ ft}^2$

Interval for x:
largest x when $y = 0$
$\Rightarrow 4x = 400$
$\Rightarrow x = 100.$

use interval $[0,100]$.

Test: $A(0) = 0$
$A(50) = 5000$
$A(100) = 0.$

This confirms our critical # yields the maximum area

20. Two birds leave the same tree at different times, one traveling due East, and the other traveling due North. At 2pm the eastbound bird is traveling at 10 mph and is 30 miles from the tree, while the northbound bird is traveling at 25 mph and is 40 miles from the tree. At what rate is the distance between the birds increasing?

Possibilities:

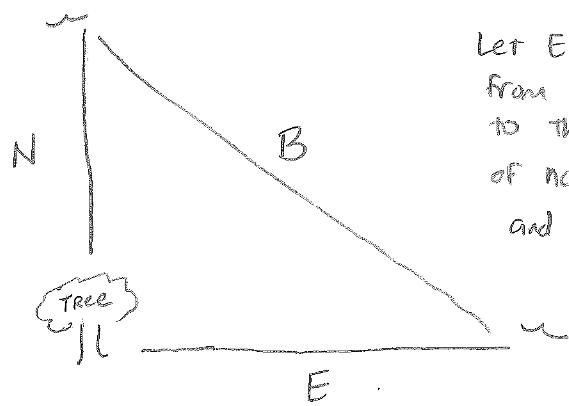
(a) $5\sqrt{29}$ mph

(b) 50 mph

(c) 2600 mph

(d) 35 mph

(e) 26 mph



Let E be the distance from the eastbound bird to the tree, N the dist. of northbound bird from tree, and B the distance between the birds.

at 2pm, if $E = 30$ and $N = 40$, then $B^2 = 30^2 + 40^2$
$\Rightarrow B = 50.$

$$E^2 + N^2 = B^2$$

$$2E \cdot \frac{dE}{dt} + 2N \cdot \frac{dN}{dt} = 2B \cdot \frac{dB}{dt}$$

$$2(30)(10) + 2(40)(25) = 2(50) \cdot \frac{dB}{dt}$$

$$100 \frac{dB}{dt} = 2600 \Rightarrow \frac{dB}{dt} = \frac{2600}{100} = 26 \text{ miles/hour}$$

Some Formulas

1. **Summation formulas:**

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

2. **Areas:**

   (a) Triangle $A = \frac{bh}{2}$

   (b) Circle $A = \pi r^2$

   (c) Rectangle $A = lw$

   (d) Trapezoid $A = \frac{h_1 + h_2}{2} b$

3. **Volumes:**

   (a) Rectangular Solid $V = lwh$

   (b) Sphere $V = \frac{4}{3}\pi r^3$

   (c) Cylinder $V = \pi r^2 h$

   (d) Cone $V = \frac{1}{3}\pi r^2 h$

4. **Distance:**

   (a) Distance between (x_1, y_1) and (x_2, y_2)

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$