MA123, Chapter 4: Computing Some Derivatives (pp. 69-82, Gootman)

`Chapter Goals:`	• Understand the derivative as the slope of the tangent line at a point. • Investigate further the notions of continuity and differentiability. • Use the definition to calculate some derivatives. • Use the definition to approximate some derivatives.
`Assignments:`	`Assignment 06` `Assignment 07`

In this chapter we explore further the relation between the derivative and the equation of the tangent line at a point. Then we learn how to compute the derivative of some functions using the definition of the derivative. One reason for doing this is to convince you that the rules and formulas for derivatives are not magical. They have a solid foundation and can be explained with just a little bit of effort. Learning should not just be a matter of memorizing mysterious formulas but it should rather be a matter of understanding them.

We start by recalling the following facts that we encountered in Chapter 2:

► **Basic facts about derivatives:** The instantaneous rate of change of a function f with respect to x at a general point x is called the *derivative of* f *at* x and is denoted with $f'(x)$:

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.}$$

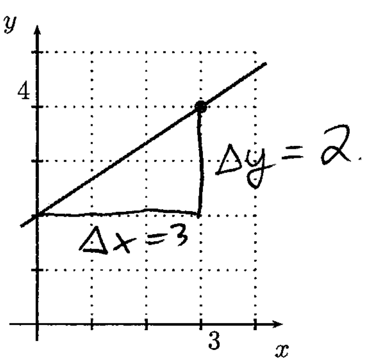
For a given value x_0, *the derivative of* f *at* x_0, *namely* $f'(x_0)$, *gives the slope of the tangent line to the graph of* f *at the point* $(x_0, f(x_0))$. Thus, the equation of the tangent line to such a point is given by the formula

$$\boxed{y = f(x_0) + f'(x_0)(x - x_0).}$$

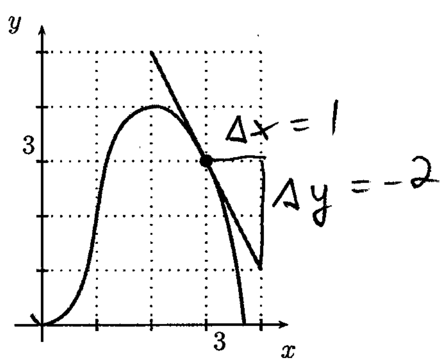
► **Tangent lines, continuity and differentiability:** In the following problems we practice computing equations of tangent lines. Also, we investigate further the notions of continuity and differentiability of a function at a point. Please refer back to Chapter 3 for the corresponding definitions.

Example 1: The graph of a function $h(x)$ and the coordinates of a point $(x_0, h(x_0))$ on the graphs of h are given below. Find $h'(x_0)$ by analyzing the graph.

h'(3) = Slope of tangent line



$h'(3) \stackrel{?}{=}$ $\frac{2}{3}$



$h'(3) \stackrel{?}{=}$ $-2.$

Note: In the following problems you can use the fact that the derivative of $f(x) = ax^2 + bx + c$ is $f'(x) = 2ax + b$. (See the calculation carried out in Chapter 2, Example 15.)

Example 2: Consider the function $f(x) = 3x^2 - 6x - 10$. Write the equation of the tangent line to the graph of f at $x = -2$ in the form $y = mx + b$, for appropriate constants m and b.

$f'(x) = 2\cdot 3x - 6 = 6x - 6$.

Slope $= f'(-2) = 6(-2) - 6 = -18$.

Point: $(-2, f(-2)) = (-2, 3(-2)^2 - 6(-2) - 10) = (-2, 14)$.

Line: $y - 14 = -18(x - (-2))$

$\Rightarrow y = -18x - 36 + 14$

$\boxed{y = -18x - 22}$

Example 3: Consider the function $g(x) = -3x^2 + 7x - 6$. Write an equation of the tangent line to the graph of g at $x = 1$. For which values of y_1 and y_2 does this tangent line go through the points $(-1, y_1)$ and $(4, y_2)$?

Slope $= g'(1) = (2)(-3)\cdot 1 + 7 = 1$

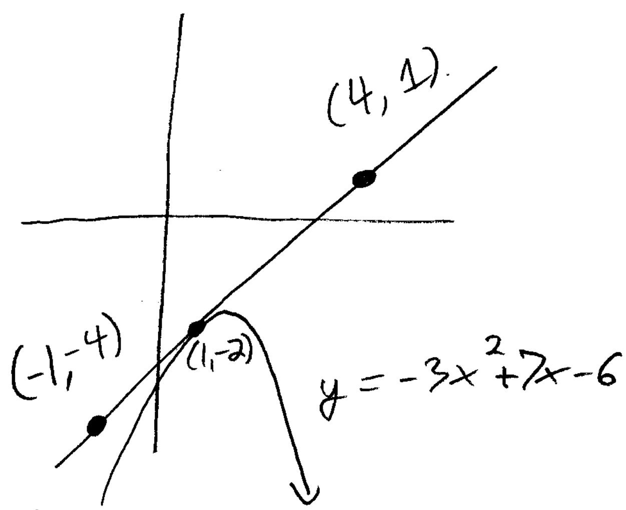
Point $= (1, g(1)) = (1, -3\cdot 1^2 + 7\cdot 1 - 6) = (1, -2)$

Tangent Line $\Rightarrow y - (-2) = 1(x - 1)$

$\Rightarrow y = x - 3$

Put $x = -1$: $y_1 = -1 - 3 = -4$

Put $x = 4$: $y_2 = 4 - 3 = 1$



Example 4: Suppose that the equation of the tangent line to the graph of the function $f(x) = \sqrt{x} + a$ at $x = 16$ is given by $y = mx + 5$. Find a and m. (**Hint:** You may use $f'(x) = \frac{1}{2\sqrt{x}}$.)

Slope $= f'(16) = \frac{1}{2\sqrt{16}} = \frac{1}{2\cdot 4} = \boxed{\frac{1}{8} = m}$

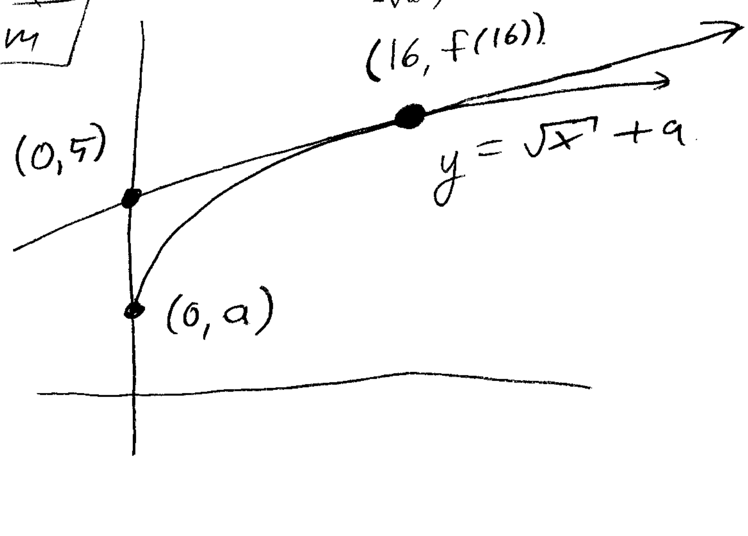
So tangent line is $y = \frac{1}{8}x + 5$.

Now, tangent line & curve agree at $x = 16$, so

$f(16) = \sqrt{16} + a = 4 + a$

$= (\frac{1}{8})16 + 5 = 7$

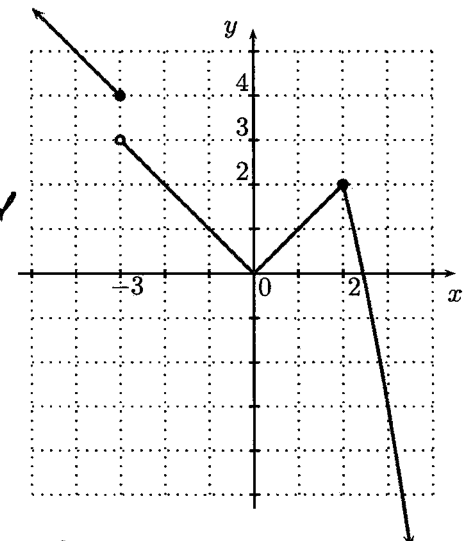
So $4 + a = 7 \Rightarrow \boxed{a = 3}$



Corners & jumps at $x = -3, 0, 2$, so not differentiable at $x = -3, 0, 2$

Example 5: Determine the x values where the derivative of the function is not defined (that is the points where the function is not differentiable). Is the function continuous at those points?

$$g(x) = \begin{cases} -x+1 & \text{if } x \le -3 \\ |x| & \text{if } -3 < x < 2 \\ -x^2+6 & \text{if } x \ge 2 \end{cases}$$



Discontinous at $x=-3$, since

$\lim_{x\to -3^-} g(x) = \lim_{x\to -3^-} -(-3)+1 = 4$

$\lim_{x\to -3^+} g(x) = \lim_{x\to -3^+} |x| = 3$,

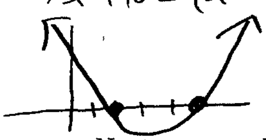
one sided limits not $=$

Continuous at $x = 2 + 0$,

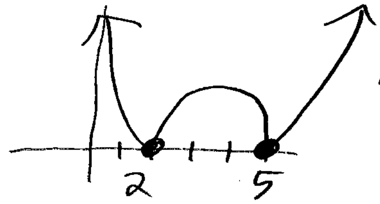
$\lim_{x\to 0} |x| = 0$, $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} |x| = 2 = \lim_{x\to 2^+} -x^2+6$.

Example 6: Determine the x values where the derivative of $h(x) = |x^2 - 7x + 10|$ is not defined. Is $h(x)$ continuous at those points? (**Hint:** first draw the graph of the equation $y = x^2 - 7x + 10$ and then draw the graph of the function h.)

$x^2 - 7x + 10 = (x-5)(x-2)$,



Now, with absolute values



Continuous everywhere, Not differentiable at $x = 2, 5$.

Next, we use the definition of the derivative to learn how to differentiate functions of the following types:

$$f(x) = (x+\alpha)^2 \qquad f(x) = \frac{1}{x+\alpha} \qquad f(x) = \sqrt{x+\alpha} \qquad f(x) = (x+\alpha)^3$$

where α is an arbitrary real number. For each type of function, the calculation of the limit has to be treated with a different technique.

Example 7: Let $f(x) = (x+4)^2$.

- Find constants A, B, and C such that $\dfrac{f(x+h) - f(x)}{h} = Ax + Bh + C$.
- Show that the derivative of f is given by the expression $f'(x) = 2x + 8 = 2(x+4)$.
- Find $f'(5)$. Write the equation of the tangent line to the graph of f at $x = 5$ in the form $y = mx + b$.

$f(x+h) = (x+h+4)^2 = ((x+4)+h)^2 = (x+4)^2 + 2(x+4)h + h^2$

so $\dfrac{f(x+h)-f(x)}{h} = \dfrac{\cancel{(x+4)^2} + 2(x+4)h + h^2 - \cancel{(x+4)^2}}{h} = \dfrac{(2(x+4)+h)\cancel{h}}{\cancel{h}} = 2x+8+h$

$A=2,\ B=1,\ C=8$

$f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (2x+8+h) = 2x+8+0 = 2x+8$.

$f'(5) = 2\cdot 5 + 8 = 18$, $=$ Slope,

Point $= (5, (5+4)^2) = (5, 81)$

$y - 81 = 18(x-5)$

$\Rightarrow y = 18x - 9$

Note: Expanding the binomial $(x+4)^2$ in the expression for the function f of Example 6 yields that $f(x) = x^2 + 8x + 16$. Hence the result $f'(x) = 2x + 8$ also follows from the calculation carried out in Chapter 2, Example 15.

Example 8: Let $f(x) = \dfrac{1}{x+3}$.

- Find constants A, B, C, and D such that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{A}{(x+B)(x+Ch+D)}$.
- Show that the derivative of f is given by the expression $f'(x) = \dfrac{-1}{(x+3)^2} = -(x+3)^{-2}$.
- Find $f'(5)$.

$f(x+h) = \dfrac{1}{x+h+3}$, so $\dfrac{f(x+h)-f(x)}{h} = \dfrac{1}{h}\left[\dfrac{1}{x+h+3} - \dfrac{1}{x+3}\right] = \dfrac{1}{h}\left[\dfrac{(x+3)-(x+h+3)}{(x+3)(x+h+3)}\right]$

$= \dfrac{1}{h}\cdot\left[\dfrac{-h}{(x+3)(x+h+3)}\right] = \dfrac{-1}{(x+3)(x+h+3)}$, $A = -1$, $B = 3$, $C = 1$, $D = 3$

$f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \dfrac{-1}{(x+3)(x+3)} = \dfrac{-1}{(x+3)^2}$

$f'(5) = \dfrac{-1}{(5+3)^2} = \dfrac{-1}{64}$.

$A = 1$, $B = 1$, $C = 1$, $D = -2$.

Example 9: Let $f(x) = \sqrt{x-2}$.

- Find constants A, B, C, and D such that $\dfrac{f(x+h) - f(x)}{h} = \dfrac{A}{\sqrt{Bx + Ch + D} + \sqrt{x-2}}$.
- Show that the derivative of f is given by the expression $f'(x) = \dfrac{1}{2\sqrt{x-2}} = \dfrac{1}{2}(x-2)^{-1/2}$.
- Find $f'(6)$ and $f'(11)$.

multiply by conjugate

$\dfrac{f(x+h)-f(x)}{h} = \dfrac{\sqrt{x+h-2} - \sqrt{x-2}}{h} = \left(\dfrac{\sqrt{x+h-2} - \sqrt{x-2}}{h}\right)\left(\dfrac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}\right)$

$= \dfrac{(\cancel{x}+h-\cancel{2})-(\cancel{x}-\cancel{2})}{h(\sqrt{x+h-2}+\sqrt{x-2})} = \dfrac{\cancel{h}}{\cancel{h}(\sqrt{x+h-2}+\sqrt{x-2})} = \dfrac{1}{\sqrt{x+h-2}+\sqrt{x-2}}$

$f'(x) = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{x+h-2}+\sqrt{x-2}} = \dfrac{1}{\sqrt{x+0-2}+\sqrt{x-2}} = \dfrac{1}{2\sqrt{x-2}}$

$f'(6) = \dfrac{1}{2\sqrt{6-2}} = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$

$f'(11) = \dfrac{1}{2\sqrt{11-2}} = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$

Special product formulas: The powers of certain binomials occur so frequently that we should memorize the following formulas. We can verify them by performing the multiplications.

If A and B are any real numbers or algebraic expressions, then:

(1.) $(A+B)^2 = A^2 + 2AB + B^2$

(2.) $(A-B)^2 = A^2 - 2AB + B^2$

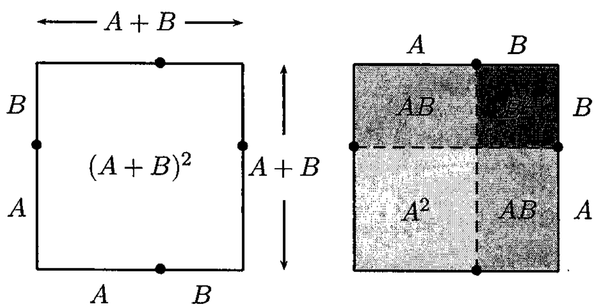
(3.) $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

(4.) $(A-B)^3 = A^3 - 3A^2B + 3AB^2 - B^3$

Visualizing a formula:

Many of the special product formulas can be seen as geometrical facts about length, area, and volume. The ancient Greeks always interpreted algebraic formulas in terms of geometric figures.

For example, the figure below



$$(A+B)^2 = A^2 + 2AB + B^2$$

shows how the formula for the square of a binomial (formula **1**) can be interpreted as a fact about areas of squares and rectangles.

Pascal's triangle: The coefficients (without sign) of the expansion of a binomial of the form $(a \pm b)^n$ can be read off the n-th row of the following 'triangle' named **Pascal's triangle** (after Blaise Pascal, a 17th century French mathematician and philosopher).

To build the triangle, start with '1' at the top, then continue placing numbers below it in a triangular way. Each number is simply obtained by adding the two numbers directly above it.

1	$n = 0$
1 1	$n = 1$
1 2 1	$n = 2$
1 3 3 1	$n = 3$
1 6 4 1	$n = 4$
1 5 10 10 5 1	$n = 5$
⋮	⋮

Example 10: Let $g(x) = (x-4)^3$.

- Find constants A, B, C, D, E and F such that $\dfrac{g(x+h)-g(x)}{h} = Ax^2 + Bx + C + Dxh + Eh + Fh^2$.
- Show that the derivative of g is given by the expression $g'(x) = 3(x-4)^2 = 3x^2 - 24x + 48$.
- Find $g'(6)$ and $g'(-1)$.

$$\frac{g(x+h)-g(x)}{h} = \frac{(x+h-4)^3-(x-4)^3}{h} = \frac{((x-4)+h)^3-(x-4)^3}{h}$$

$$= \frac{(x-4)^3+3(x-4)^2h+3(x-4)h^2+h^3-\cancel{(x-4)^3}}{h} = \frac{\cancel{h}(3(x-4)^2+3(x-4)h+h^2)}{\cancel{h}}$$

$$= 3(x-4)^2+3(x-4)h+h^2 = 3x^2-24x+48+3xh-12h+h^2.$$

$A=3$, $B=-24$, $C=48$, $D=3$, $E=-12$, $F=1$.

$$g'(x) = \lim_{h\to 0}\left(3(x-4)^2+3(x-4)h+h^2\right) = 3(x-4)^2$$

$g'(6) = 3(6-4)^2 = 12$, $\quad g'(-1) = 3(-1-4)^2 = 75$

$\boxed{\textbf{Example 11:}}$ If $f(x) = \dfrac{-2}{x-3}$, then $\dfrac{f(x+h)-f(x)}{h} = \dfrac{A}{(x-3)(x+Bh+C)}$. Find A, B, and C.

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\frac{-2}{x+h-3} - \frac{-2}{x-3}\right] = \frac{-2}{h}\left[\frac{1}{x+h-3} - \frac{1}{x-3}\right] = \frac{-2}{h}\left[\frac{(x-3)-(x+h-3)}{(x-3)(x+h-3)}\right]$$

$$= \frac{-2}{\cancel{h}}\left[\frac{-\cancel{h}}{(x-3)(x+h-3)}\right] = \frac{2}{(x-3)(x+h-3)}$$

so $A = 2$, $B = 1$, $C = -3$.

$\boxed{\textbf{Example 12:}}$ Suppose that $\dfrac{f(x+h)-f(x)}{h} = \dfrac{-2h(x+2)-h^2}{h(x+h+2)^2(x+2)^2}$.
Find the slope m of the tangent line at $x = 1$.

$$m = f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{-2h(1+2)-h^2}{h(1+h+2)^2(1+2)^2}$$

$$= \lim_{h\to 0} \frac{-6h-h^2}{h(3+h)^2 \cdot 9} = \lim_{h\to 0} \frac{-\cancel{h}(6+h)}{\cancel{h}\cdot 9\cdot(3+h)^2}$$

$$= \lim_{h\to 0} \frac{-(6+h)}{9(3+h)^2} = \frac{-6}{9\cdot 3^2} = -\frac{2}{27}.$$

$\boxed{\textbf{Example 13:}}$ Suppose that

$$\frac{f(x+h)-f(x)}{h} = 3x + 2h - 1 \qquad \text{and} \qquad f(1) = 4.$$

Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$.

Point: $(1, 4)$.

Slope $= f'(1) = \lim_{h\to 0} \dfrac{f(1+h)-f(1)}{h} = \lim_{h\to 0} (3\cdot 1 + 2h - 1)$

$= 3 + 2\cdot 0 - 1 = 2$

Tangent line: $y - 4 = 2(x-1)$

$\Rightarrow y = 2x + 2.$

► **Approximating a derivative:** Considering how much effort was required to compute the derivative of a seemingly simple function like $g(x) = \sqrt{x-2}$, you may think that it will be almost impossible to compute the derivative of functions like $G(x) = e^{-x^2}$ or $h(x) = (x^2+17)^9$. In the next two chapters we will learn algebraic formulas which will help compute derivatives of lots of functions. We end this chapter by suggesting an alternative method for finding derivatives of more complex functions: numeric approximation.

Example 14: Let $f(x) = (x+7)^5$. Approximate $f'(2)$. ≈ 32805

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{f(2+h)-f(2)}{h} = \frac{(h+9)^5 - 9^5}{h}$	32084.0551	32732.181	32797.7108	32804.271	32804.9271	32805.0729	32805.729	32812.2908	32877.981	33542.1451

Example 15: Let $g(x) = \ln(x)$. Approximate $g'(2)$. $\approx 0.5 = \frac{1}{2}$

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{g(2+h)-g(2)}{h} = \frac{\ln(2+h) - \ln(2)}{h}$	.5129	.5012	.50012	.5000125	.50000125	.49999875	.4999875	.499875	.498754	.4879.

Example 16: Let $f(x) = 3^x$. Approximate $f'(1)$. $\approx 3.2958\ldots$

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{f(1+h)-f(1)}{h} = \frac{3 \cdot (3^h - 1)}{h}$	3.1212...	3.2777987	3.29402711	3.295665...	3.295818...	3.29585	3.2960179	3.2976...	3.314...	3.48369...

Example 17: Let $g(x) = |5x|$. Approximate $g'(0)$. does not exist (one sided limits disagree)

h	-0.1	-0.01	-0.001	-0.0001	-0.00001	0.00001	0.0001	0.001	0.01	0.1
$\frac{g(0+h)-g(0)}{h} = \frac{\lvert 5h \rvert}{h}$	−1	−1	−1	−1	−1	1	1	1	1	1