Chapter Goals: In this Chapter we learn a general strategy on how to approach optimization problems, one of the main types of word problems that one usually encounters in a first Calculus course.

Assignments: Assignment 17

MAX-MIN PROBLEMS

All max-min problems ask you to find the largest or smallest value of a function on an interval. Usually, the hard part is reading the English and finding the formula for the function. Once you have found the function, then you can use the techniques from Chapter 6 to find the largest or smallest values.

▶ **Max-min guideline:** This guideline is found on pp. 131-133 of our textbook.

(1.) Read the problem quickly.
(2.) Read the problem carefully.
(3.) Define your variables. If the problem is a geometry problem, draw a picture and label it.
(4.) Determine whether you need to find the max or the min.
Determine exactly what needs to be maximized or minimized.
(5.) Write the *general* formula for what you are trying to maximize or minimize. If this formula only involves *one* variable, then skip steps **6**, **7** and **8.**
(6.) Find the relationship(s) (i.e., equation(s)) between the variables.
(7.) Do the algebra to solve for one variable in the equation(s) as a function of the other(s).
(8.) Use your formula from step **5** to rewrite the formula that you want to maximize or minimize as a function of one variable only.
(9.) Write down the interval over which the above variable can vary, for the particular word problem you are solving.
(10.) Take the derivative and find the critical points.
(11.) Use the techniques from Chapter 6 to find the maximum or the minimum.

Example 1: What is the largest possible product you can form from two non-negative numbers whose sum is 30?

let x, y be our nonnegative numbers.

We know: $x+y=30$
We want to maximize $x \cdot y$

$x+y=30$
$y=30-x$

$x \geq 0$ $\quad$ $y \geq 0$
$30-x \geq 0$
$x \leq 30$

thus we need to find a max on $[0,30]$.

$x \cdot y = x(30-x) = 30x - x^2$
let $f(x) = 30x - x^2$
then $f'(x) = 30-2x$
$f'(x)=0$ when $x=15$
So max can occur at $x=15$ or endpoints
Check: $f(0)=30(0)-0^2=0$
$f(15)=30(15)-15^2=225 \leftarrow$ max
$f(30)=30(30)-30^2=0$

$\boxed{\text{Max product occurs at } x=15}$

Example 2: Suppose the product of x and y is 26 and both x and y are positive. What is the minimum possible sum of x and y?

We know $x \cdot y = 26$ $\quad x > 0 \quad y > 0$

$x \cdot y = 26$

$y = \frac{26}{x}$

$y = \frac{26}{x} > 0$ for all $x > 0$

We need to find minimum sum $x + y$

$x + y = x + \frac{26}{x} = x + 26x^{-1}$

Let $f(x) = x + 26x^{-1}$

$f'(x) = 1 - 26x^{-2} = 1 - \frac{26}{x^2}$

f' not defined at $x = 0$

$f'(x) = 0$ when $x^2 = 26$

$x = \pm\sqrt{26}$

but since $x > 0$, we only need $x = \sqrt{26}$

min

$-$ $\quad$ $+$

$0 \quad$ test 1 $\quad \sqrt{26} \quad$ test 10

$f'(1) = 1 - \frac{26}{1^2} = 1 - 26 = -25$ "$-$"

$f'(10) = 1 - \frac{26}{10^2} = 1 - \frac{26}{100} = \frac{74}{100}$ "$+$"

minimum sum occurs at $x = \sqrt{26}$

minimum possible sum is $x + y = \sqrt{26} + \frac{26}{\sqrt{26}}$

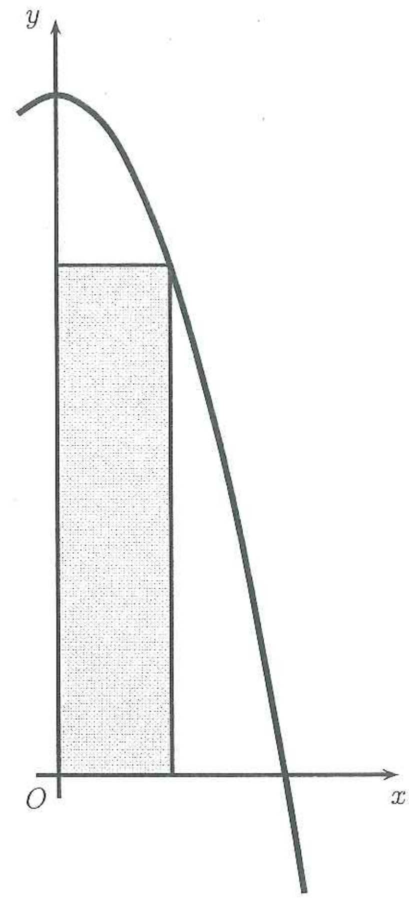
$= \sqrt{26} + \sqrt{26}$

$= \boxed{2\sqrt{26}}$

Note: An alternative wording for Example 2 above is:

"Suppose y is inversely proportional to x and the constant of proportionality equals 26. What is the minimum sum of x and y if x and y are both positive?"

Example 3: Find the area of the largest rectangle with one corner at the origin, the opposite corner in the first quadrant on the graph of the parabola $f(x) = 9 - x^2$, and sides parallel to the axes.



Need to find: Max area $A = x \cdot y = x(9 - x^2) = 9x - x^3$

because we're in the 1st quadrant; $x \geq 0$, $y \geq 0$

so $9 - x^2 \geq 0$

$9 \geq x^2$

$-3 \leq x \leq 3$ $\quad$ thus $x \in [0, 3]$

Let $f(x) = 9x - x^3$ then $f'(x) = 9 - 3x^2 = 3(3 - x^2)$

$f'(x) = 0$ when $x^2 = 3$

$x = \pm\sqrt{3}$

$-\sqrt{3}$ not in interval $[0, 3]$

So max could occur at $x = \sqrt{3}$ or endpoints 0 and 3

Check: $f(\sqrt{3}) = 9\sqrt{3} - (\sqrt{3})^3 = 6\sqrt{3}$ ← max

$f(0) = 9(0) - 0^3 = 0$

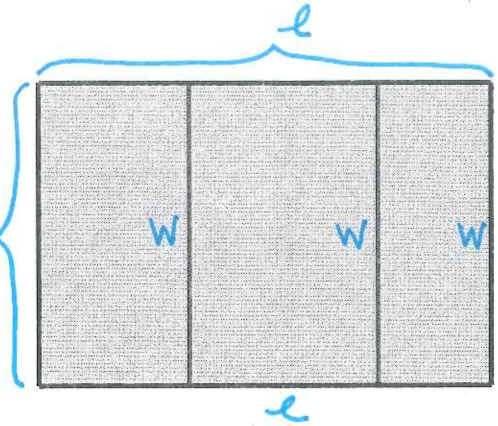
$f(3) = 9(3) - 3^3 = 0$

Max area occurs when $x = \sqrt{3}$.

if $x = \sqrt{3}$ then $y = 9 - x^2 = 9 - (\sqrt{3})^2 = 9 - 3 = 6$

So the max area is $x \cdot y = \sqrt{3} \cdot 6 = \boxed{6\sqrt{3}}$

Example 4: A farmer builds a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



We know: $2l + 4w = 500$

$l, w \geq 0$

We need: to maximize Area $A = l \cdot w$

$2l + 4w = 500$

$2l = 500 - 4w$

$l = 250 - 2w$

if $l \geq 0$

$250 - 2w \geq 0$

$250 \geq 2w$

$125 \geq w$

So $w \in [0, 125]$

$A = l \cdot w = (250 - 2w)w = 250w - 2w^2$

let $f(w) = 250w - 2w^2$

then $f'(w) = 250 - 4w$

$f'(w) = 0$ when $w = \frac{250}{4} = \frac{125}{2}$

So max could occur at $w = 0, \frac{125}{2}, 125$

(critical value or endpoints)

Check: $f(0) = 250(0) - 2(0)^2 = 0$

$f\left(\frac{125}{2}\right) = 250\left(\frac{125}{2}\right) - 2\left(\frac{125}{2}\right)^2 = \underset{max}{\underline{7812.5}}$

$f(125) = 250(125) - 2(125)^2 = 0$

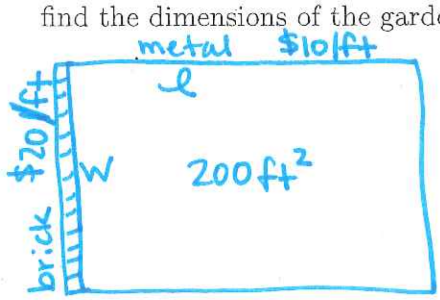
Max occurs when $w = \frac{125}{2}$ need l.

$l = 250 - 2\left(\frac{125}{2}\right) = 125$

The dimensions to maximize area are

$\boxed{l = 125 \text{ ft.} \quad w = \frac{125}{2} \text{ ft.}}$

Example 5: A landscape architect wishes to enclose a rectangular garden on one side by a brick wall costing \$20/ft and on the other three sides by a metal fence costing \$10/ft. If the area of the garden is 200 square feet, find the dimensions of the garden that minimize the cost.



We know: $A = 200 \text{ ft}^2 = l \cdot w$ and $l, w \geq 0$

We need: to minimize cost $C = \$20 \cdot w + \$10(2l + w)$

$l \cdot w = 200$

$l = \frac{200}{w}$

$\frac{200}{w} > 0$ for all $w > 0$

So $w \in (0, \infty)$

$C = 20w + 10(2l + w)$

$= 20w + 20l + 10w$

$= 30w + 20l$

$= 30w + 20\left(\frac{200}{w}\right)$

$= 30w + 4000w^{-1}$

let $f(w) = 30w + 4000w^{-1}$

then $f'(w) = 30 - 4000w^{-2} = 30 - \frac{4000}{w^2}$

$\left(30 - \frac{4000}{w^2} = 0\right) w^2$

$30w^2 - 4000 = 0$

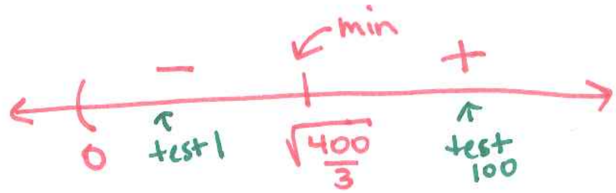
$30w^2 = 4000$

$w^2 = \frac{400}{3}$

$w = \pm\sqrt{\frac{400}{3}}$

$f'(w) = 0$ when $w = \pm\sqrt{\frac{400}{3}}$

but $-\sqrt{\frac{400}{3}}$ not in interval.



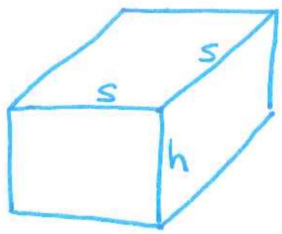
$f'(1) = 30 - \frac{4000}{1^2} = \text{"}-\text{"}$

$f'(100) = 30 - \frac{4000}{100^2} = \text{"}+\text{"}$

Min. cost occurs at $\boxed{w = \sqrt{\frac{400}{3}} \approx 11.547 \text{ ft.}}$

then $l = \frac{200}{w} = \frac{200}{\sqrt{\frac{400}{3}}} = \boxed{10\sqrt{3} \approx 17.321 \text{ ft.} = l}$

Example 6: A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$5 per square foot and the metal for the sides costs \$10 per square foot. Find the dimensions that minimize cost if the box has a volume of 24 cubic feet.



We know: $V = 24\text{ ft}^3 = s\cdot s\cdot h$ and $s, h \geq 0$

We need: minimize cost $C = \$5(2s^2) + \$10(4sh)$

$24 = s^2 h$

$h = \frac{24}{s^2} \geq 0$ when $s > 0$

then $s \in (0, \infty)$

$$C = 5(2s^2) + 10(4sh)$$
$$= 10s^2 + 40sh$$
$$= 10s^2 + 40s\left(\frac{24}{s^2}\right)$$
$$= 10s^2 + \frac{960}{s}$$

let $f(s) = 10s^2 + 960s^{-1}$

then $f'(s) = 20s - 960s^{-2} = 20s - \frac{960}{s^2}$

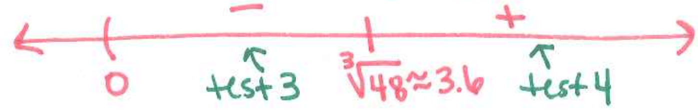
$$= \frac{20s^3 - 960}{s^2}$$

$f'(s) = 0$ when $20s^3 - 960 = 0$

$s^3 - 48 = 0$

$s^3 = 48$

$s = \sqrt[3]{48}$



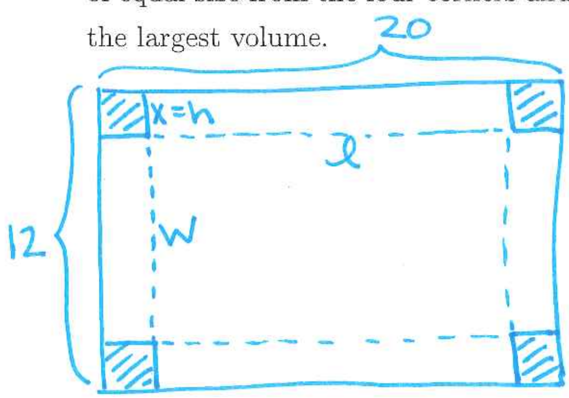
$f'(3) =$ "−"

$f'(4) =$ "+"

Min cost occurs when $\boxed{s = \sqrt[3]{48} \approx 3.634 \text{ ft.}}$

and $\boxed{h = \frac{24}{(\sqrt[3]{48})^2} \approx 1.817 \text{ ft.}}$

Example 7: An open box is to be made out of a 12-inch by 20-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. Find the dimensions of the resulting box that has the largest volume.



We need: maximize volume

$V = l\cdot w\cdot h$

$= (20-2x)(12-2x)(x)$ $\quad l, w, h \geq 0$

$l \geq 0$	$w \geq 0$	$h \geq 0$
$20 - 2x \geq 0$	$12 - 2x \geq 0$	$x \geq 0$
$20 \geq 2x$	$12 \geq 2x$	
$10 \geq x$	$6 \geq x$	

thus $x \in [0, 6]$

$$V = (20-2x)(12-2x)(x)$$
$$= (240 - 64x + 4x^2)(x)$$
$$= 240x - 64x^2 + 4x^3$$

let $V(x) = 240x - 64x^2 + 4x^3$

then $V'(x) = 240 - 128x + 12x^2$

$$= 4(60 - 32x + 3x^2)$$

$V'(x) = 0$ when $60 - 32x + 3x^2 = 0$

quadratic formula!

$$x = \frac{32 \pm \sqrt{(-32)^2 - 4(3)(60)}}{2(3)} = \frac{32 \pm \sqrt{304}}{6} = \frac{16 \pm \sqrt{76}}{3}$$

$x = \frac{16 + \sqrt{76}}{3}$ not in interval

max could occur at $x = 0, 6,$ or $\frac{16 - \sqrt{76}}{3}$

Check: $V(0) = 240(0) - 64(0^2) + 4(0^3) = 0$

$V(6) = 240(6) - 64(6^2) + 4(6^3) = 0$

$V\left(\frac{16-\sqrt{76}}{3}\right) \approx 262.68 \leftarrow$ max

Max volume occurs when $x = \frac{16 - \sqrt{76}}{3}$

dimensions are

$\boxed{15.145 \times 7.145 \times 2.427 \text{ ft}}$

Example 8: A car rental agency rents 180 cars per day at a rate of 30 dollars per day. For each 1 dollar increase in the daily rate, 5 fewer cars are rented. At what rate should the cars be rented to produce the maximum income, and what is the maximum income?

let $n = \#$ of cars rented
$p =$ price of car rental
$x = \#$ of \$1 increases in daily rate

then Income $\underline{I = n \cdot p \quad \text{maximize}}$

$$I = n \cdot p$$
$$= (180 - 5x)(30 + x)$$
$$= 5400 + 30x - 5x^2$$

$n, p \geq 0$

$180 - 5x \geq 0$
$180 \geq 5x$
$36 \geq x$

$30 + x \geq 0$
$-30 \leq x$

so $x \in [-30, 36]$

let $f(x) = 5400 + 30x - 5x^2$

then $f'(x) = 30 - 10x$

$f'(x) = 0$ when $x = 3$

max could occur at $x = -30, 3, 36$

Check: $f(-30) = 5400 + 30(-30) - 5(-30)^2 = 0$
$f(3) = 5400 + 30(3) - 5(3)^2 = 5445 \leftarrow$ max
$f(36) = 5400 + 30(36) - 5(36)^2 = 0$

Max occurs at $x = 3$

then $p = 30 + 3 = \boxed{\$33\ /\text{day}}$

and Income $= f(3) = \boxed{\$5445}$