

Exponential Models of Population Growth

The formula for population growth of several species is the same as that for continuously compounded interest. In fact in both cases the rate of growth r of a population (or an investment) per time period is proportional to the size of the population (or the amount of an investment).

Exponential Growth Model

If n_0 is the initial size of a population that experiences **exponential growth**, then the population $n(t)$ at time t increases according to the model

$$n(t) = n_0 e^{rt}$$

where r is the relative rate of growth of the population (expressed as a proportion of the population).

Remark:

Biologists sometimes express the growth rate in terms of the **doubling-time** h , the time required for the population to double in size: $r = \frac{\ln 2}{h}$.

Proof: Indeed, from

$$2n_0 = n(h) = n_0 e^{rh}$$

we obtain

$$2 = e^{rh} \rightsquigarrow \ln 2 = rh \rightsquigarrow r = \frac{\ln 2}{h}.$$

Using the doubling-time h , we can also rewrite $n(t)$ as:

$$\boxed{n(t)} = n_0 e^{rt} = n_0 e^{\frac{\ln 2}{h} t} = n_0 e^{(t/h) \cdot \ln 2} = n_0 e^{\ln(2^{t/h})} = \boxed{n_0 2^{t/h}}.$$

Radioactive Decay

Radioactive substances decay by spontaneously emitting radiations. Also in this situation, the rate of decay is proportional to the mass of the substance and is independent of environmental conditions. This is analogous to population growth, except that the mass of radioactive material *decreases*.

Radioactive Decay Model

If m_0 is the initial mass of a radioactive substance then the mass $m(t)$ remaining at time t is modeled by the function

$$m(t) = m_0 e^{-rt}$$

where r is the relative rate of decay of the radioactive substance.

Remark:

Physicists sometimes express the rate of decay in terms of the **half-life** h , the time required for half the mass to decay: $r = \frac{\ln 2}{h}$.

Proof: Indeed, from

$$\frac{1}{2}m_0 = m(h) = m_0e^{-rh}$$

we obtain

$$\frac{1}{2} = e^{-rh} \rightsquigarrow \ln \frac{1}{2} = -rh \rightsquigarrow -\ln 2 = -rh \rightsquigarrow r = \frac{\ln 2}{h}.$$

Using the half-time h , we can also rewrite $m(t)$ as:

$$m(t) = m_0e^{-rt} = m_0e^{-\frac{\ln 2}{h}t} = m_0e^{(-t/h) \cdot \ln 2} = m_0e^{\ln(2^{-t/h})} = m_0\left(\frac{1}{2}\right)^{t/h}$$

Example 6: (Online Homework HW03, # 8)

A town has population 750 people at year $t = 0$.

Write a formula for the population, P , in year t if the town

- (a) Grows by 70 people per year
- (b) Grows by 12% per year
- (c) Grows at a continuous rate of 12% per year.
- (d) Shrinks by 14 people per year.
- (e) Shrinks by 4% per year.
- (f) Shrinks at a continuous rate of 4% per year.

Example 7 (Frog Population):

The frog population in a small pond grows exponentially. The current population is 85 frogs, and the relative growth rate is 18% per year.

- (a) Which function models the population after t years?
- (b) Find the projected frog population after 3 years.
- (c) When will the frog population reach 600?
- (d) When will the frog population double?

Example 8: (Online Homework HW03, # 14)

Assume that the number of bacteria follows an exponential growth model: $P(t) = P_0 e^{kt}$. The count in the bacteria culture was 100 after 15 minutes and 1800 after 35 minutes.

- (a) What was the initial size of the culture?
- (b) Find the population after 105 minutes.
- (c) How many minutes after the start of the experiment will the population reach 14,000?

Example 9:

The mass $m(t)$ remaining after t days from a 40-g sample of thorium-234 is given by:

$$m(t) = 40e^{-0.0277 t}.$$

- (a) How much of the sample will be left after 60 days?
- (b) After how long will only 10-g of the sample remain?

From Neuhauser's Textbook, p. 27

[...] Carbon 14 is formed high in the atmosphere. It is radioactive and decays into nitrogen (N^{14}).

There is an equilibrium between atmospheric carbon 12 (C^{12}) and carbon 14 (C^{14}) — a ratio that has been relatively constant over a fairly long period.

When plants capture carbon dioxide (CO_2) molecules from the atmosphere and build them into a product (such as cellulose), the initial ratio of C^{14} to C^{12} is the same as that in the atmosphere.

Once the plants die, however, their uptake of CO_2 ceases, and the radioactive decay of C^{14} causes the ratio of C^{14} to C^{12} to decline.

Because the law of radioactive decay is known, the change in ratio provides an accurate measure of the time since the plants death.

Example 10: (Neuhauser, Problem # 64, p.37)

The half-life of C^{14} is 5730 years. Suppose that wood found at an archeological excavation site contains about 35% as much C^{14} (in relation to C^{12}) as does living plant material.

Determine when the wood was cut.

Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings, provided that the temperature difference is not too large.

Using Calculus, the following model can be deduced from this law:

The Model

If D_0 is the initial temperature difference between an object and its surroundings, and if its surroundings have temperature T_S , then the temperature of the object at time t is modeled by the function

$$T(t) = T_S + D_0 e^{-kt}$$

where k is a positive constant that depends on the object.

Example 11 (Cooling Turkey):

A roasted turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F .

- (a) If the temperature of the turkey is 150°F after half an hour, what is its temperature after 45 minutes?
- (b) When will the turkey cool to 100°F ?

Interested in Forensic Pathology?

Newton's Law of Cooling is used in **homicide investigations** to determine the time of death. Immediately following death, the body begins to cool (its normal temperature is 98.6°F). It has been experimentally determined that the constant in Newton's Law of Cooling is $k \approx 0.1947$, assuming time is measured in hours.

MA 137 — Calculus 1 for the Life Sciences
Semilog and Double Log Plots
(Section 1.3)

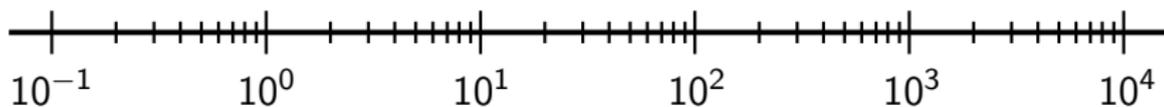
Alberto Corso
(alberto.corso@uky.edu)

Department of Mathematics
University of Kentucky

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Logarithmic Scales

- When a physical quantity varies over a very large range, it is often convenient to take its logarithm in order to have a more manageable set of numbers.
- Quantities that are measured on logarithmic scales include
 - acidity of a solution (the **pH scale**),
 - earthquake intensity (Richter scale),
 - loudness of sounds (decibel scale),
 - light intensity,
 - information capacity,
 - radiation.
- In such cases, the equidistant marks on a logarithmic scale represent consecutive powers of 10.



The pH Scale

Chemists measured the acidity of a solution by giving its hydrogen ion concentration until Sorensen, in 1909, defined a more convenient measure:

$$\text{pH} = -\log[H^+]$$

where $[H^+]$ is the concentration of hydrogen ions measured in moles per liter (M).

Solutions are defined in terms of the pH as follows:

those with $\text{pH} = 7$ (or $[H^+] = 10^{-7}M$) are *neutral*,

those with $\text{pH} < 7$ (or $[H^+] > 10^{-7}M$) are *acidic*,

those with $\text{pH} > 7$ (or $[H^+] < 10^{-7}M$) are *basic*.

Example 1 (Finding pH):

The hydrogen ion concentration of a sample of each substance is given. Calculate the pH of the substance.

(a) Lemon juice: $[H^+] = 5.0 \times 10^{-3}M$

(b) Tomato juice: $[H^+] = 3.2 \times 10^{-4}M$

(c) Seawater: $[H^+] = 5.0 \times 10^{-9}M$

Example 2 (Ion Concentration):

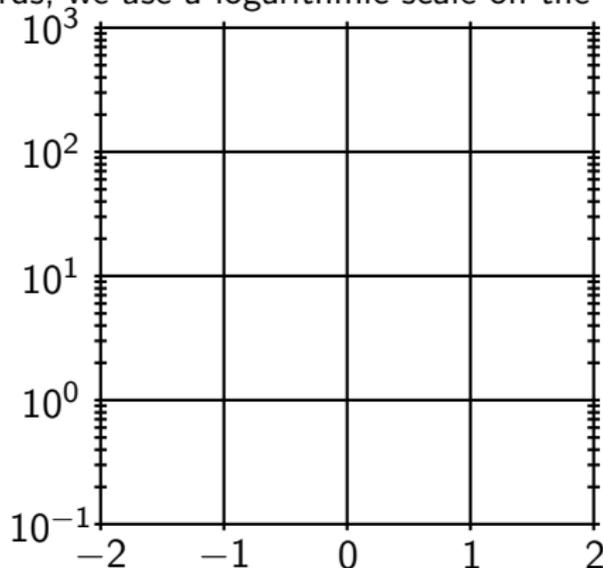
Calculate the hydrogen ion concentration of each substance from its pH reading.

(a) Vinegar: $\text{pH} = 3.0$

(b) Milk: $\text{pH} = 6.5$

Semilog Plots

- In biology its common to use a semilog plot to see whether data points are appropriately modeled by an exponential function.
- This means that instead of plotting the points (x, y) , we plot the points $(x, \log y)$.
- In other words, we use a logarithmic scale on the vertical axis.



Graphs for a Science article

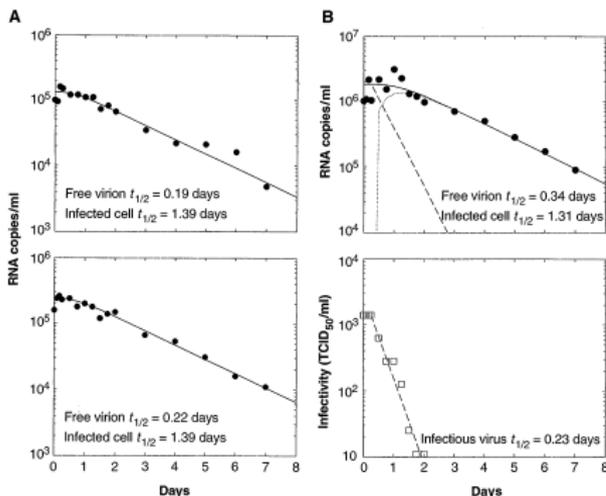


Fig. 1. (A) Plasma concentrations (copies per milliliter) of HIV-1 RNA (circles) for two representative patients (upper panel, patient 104; lower panel, patient 107) after zidovudine treatment was begun on day 0. The theoretical curve (solid line) was obtained by nonlinear least squares fitting of Eq. 6 to the data. The parameters c (virion clearance rate), δ (rate of loss of infected cells), and V_0 (initial viral load) were simultaneously estimated. To account for the pharmacokinetic delay, we assumed $t = 0$ in Eq. 6 to correspond to the time of the pharmacokinetic delay (if measured) or selected 2, 4, or 6 hours as the best-fit value (see Table 1). The logarithm of the experimental data was fitted to the logarithm of Eq. 6 by a nonlinear least squares method with the use of the subroutine DNLS1 from the Common Los Alamos Software Library, which is based on a finite difference Levenberg-Marquardt algorithm. The best fit, with the smallest sum of squares per data point, was chosen after eliminating the worst outlying data point for each patient with the use of the jack-knife method. (B) Plasma concentrations of HIV-1 RNA (upper panel; circles) and the plasma infectivity titer (lower panel; squares) for patient 105. (Top panel) The solid curve is the best fit of Eq. 6 to the RNA data; the dotted line is the curve of the noninfectious pool of virions, $V_{inf}(t)$; and the dashed line is the curve of the infectious pool of virions, $V_i(t)$. (Bottom panel) The dashed line is the best fit of the equation for $V_i(t)$ to the plasma infectivity data. TCID₅₀, 50% tissue culture infectious dose.

The graphs are taken from the article

HIV-1 Dynamics in Vivo: Virion Clearance Rate, Infected Cell Life-Span, and Viral Generation Time,

by Alan S. Perelson, Avidan U. Neumann, Martin Markowitz, John M. Leonard and David D. Ho,

Science, New Series, Vol. 271, No. 5255 (Mar. 15, 1996), pp. 1582-1586.

David Ho was Time magazine's 1996 Man of the Year.

How to Read a Semilog Plot

You need remember is that the log axis runs in exponential cycles.

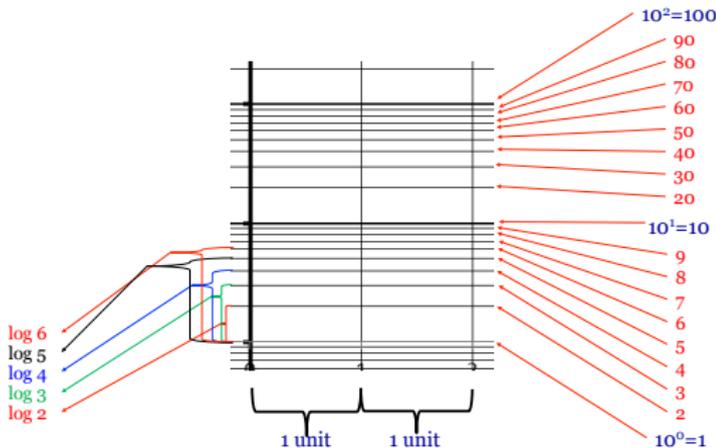
Each cycle runs linearly in 10's but the increase from one cycle to another is an increase by a factor of 10.

So within a cycle you would have a series of: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 (this could also be 0.1-1, etc.).

The next cycle begins with 10 and progresses as 20, 30, 40, 50, 60, 70, 80, 90, 100.

The cycle after that would be 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000.

Below is a picture of semilog graph paper.



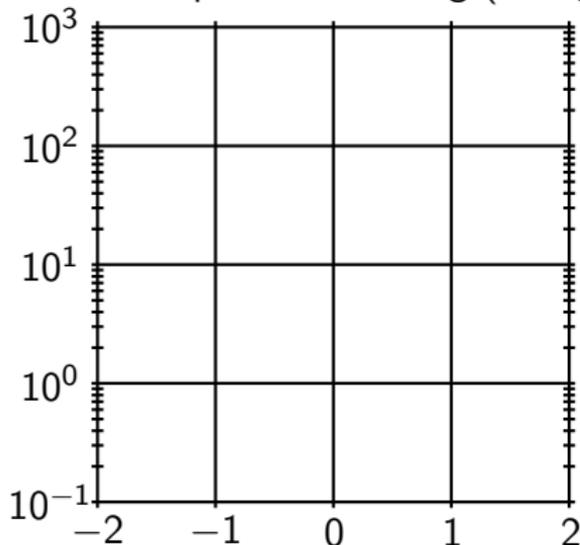
Number	log
100	2
90	1.9542
80	1.9031
70	1.8451
60	1.7782
50	1.6990
40	1.6021
30	1.4771
20	1.3010
10	1
9	0.9542
8	0.9031
7	0.8451
6	0.7782
5	0.6990
4	0.6021
3	0.4771
2	0.3010
1	0.0000

Example 3:

Suppose that x and y are related by the expression

$$y = 4 \cdot 10^{-x/2} \quad [= 4 \cdot (10^{-1/2})^x = 4 \cdot (0.316)^x].$$

Use a logarithmic transformation to find a linear relationship between the given quantities and graph the resulting linear relationship in the semilog (or log-linear) plot.



Lines in Semilog Plots

- If we start with an **exponential function** of the form $y = a \cdot b^x$ and take logarithms of both sides, we get

$$\log y = \log(a \cdot b^x) = \log a + \log b^x$$

$$\log y = \log a + x \log b$$

If we let $Y = \log y$, $M = \log b$, and $B = \log a$, then we obtain

$$Y = B + Mx,$$

i.e., the equation of a line with slope M and Y -intercept B .

- So if we obtain experimental data that we suspect might possibly be exponential, then we could graph a semilog scatter plot and see if it is approximately linear.

Example 4:

When $\log y$ is graphed as a function of x , a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (0, 40) \quad (x_2, y_2) = (2, 600)$$

on a log-linear plot. Determine the functional relationship between x and y . (**Note:** The original x - y coordinates are given.)

Example 5: (Problem # 46, Section 1.3, p. 53)

When $\log y$ is graphed as a function of x , a straight line results. Graph the straight line given by the following two points

$$(x_1, y_1) = (1, 4) \quad (x_2, y_2) = (6, 1)$$

on a log-linear plot. Determine the functional relationship between x and y . (**Note:** The original x - y coordinates are given.)

Example 6: (Problem # 52, Section 1.3, p. 53)

Consider the relationship $y = 6 \times 2^{-0.9x}$ between the quantities x and y . Use a logarithmic transformation to find a linear relationship of the form

$$Y = mx + b$$

between the given quantities.