# MA 137 – Calculus 1 with Life Science Applications
# The Chain Rule and Higher Derivatives
(Section 4.4)

**Alberto Corso**
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

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## The Chain Rule

**Theorem**

*If $g$ is differentiable at $x$ and $f$ is differentiable at $y = g(x)$, then the composite function $(f \circ g)(x) = f[g(x)]$ is differentiable at $x$, and the derivative is given by*

$$(f \circ g)'(x) = f'[g(x)] \cdot g'(x)$$

- The proof of the theorem is on p. 164 of the Neuhauser's textbook.
- The function $g$ is the inner function; the function $f$ is the outer function.
- The expression $f'[g(x)] \cdot g'(x)$ thus means that we need to find the derivative of the outer function, evaluated at $g(x)$, and the derivative of the inner function, evaluated at $x$, and then multiply the two together.
- A special case of the chain rule is called the **power chain rule**:

If $\quad y = [f(x)]^n \quad$ then $\quad \dfrac{dy}{dx} = n[f(x)]^{n-1} \cdot f'(x)$





## The Chain Rule in Leibniz Notation

The derivative of $f \circ g$ can be written in Leibniz notation.

If we set $u = g(x)$, then

$$\begin{aligned} \frac{d}{dx}[(f \circ g)(x)] &= \frac{d}{dx} f[g(x)] \\ &\overset{u=g(x)}{=} \frac{d}{dx} f(u) \\ &= \frac{df}{du} \cdot \frac{du}{dx} \end{aligned}$$

This form of the chain rule emphasizes that, in order to differentiate $f \circ g$, we multiply the derivative of the outer function and the derivative of the inner function, the former evaluated at $u$, the latter at $x$.





## Example 1: (Online Homework HW13, # 3)

Let $F(x) = f(f(x))$ and $G(x) = (F(x))^2$ and suppose that

$$f(5) = 3 \qquad f(7) = 5 \qquad f'(5) = 8 \qquad f'(7) = 13$$

Find $F'(7)$ and $G'(7)$.

## Example 2: (Online Homework HW13, # 6)

Let $f(x) = \dfrac{9}{(2x^2 - 3x + 6)^4}$. Find $f'(x)$.





## The Quotient Rule Using the Chain Rule

We can prove quotient rule using the product and (power) chain rules. Treat the quotient $f/g$ as a product of $f$ and the reciprocal of $g$. I.e.,

$$\frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}.$$

Next, apply the product rule

$$\left(\frac{f(x)}{g(x)}\right)' = [f(x) \cdot g(x)^{-1}]' = f'(x) \cdot g(x)^{-1} + f(x) \cdot [g(x)^{-1}]'$$

and apply the (power) chain rule to find $[g(x)^{-1}]'$. We obtain

$$= f'(x) \cdot [g(x)^{-1}] + f(x) \cdot [(-1)g(x)^{-2} \cdot g'(x)].$$

Finish by writing the expression with a common denominator of $[g(x)]^2$

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}.$$





## Example 3: (Neuhauser, Example # 5, p. 161)

Find the derivative of $h(x) = \left(\dfrac{x}{x+1}\right)^2$.





## Example 4: (Neuhauser, Problem # 32, p. 172)

Differentiate $g(N) = \dfrac{N}{(k + bN)^3}$ with respect to $N$.

Assume that $b$ and $k$ are positive constants.

## Example 5: (Neuhauser, Problem # 39, p. 172)

Find the derivative of

$$\frac{[f(x)]^2}{g(2x) + 2x}$$

assuming that $f$ and $g$ are both differentiable functions.





## Higher Derivatives

- The derivative of a function $f$ is itself a function. We refer to this derivative as the **first derivative**, denoted $f'$. If the first derivative exists, we say that the function is once differentiable.
- Given that the first derivative is a function, we can define its derivative (where it exists). This derivative is called the **second derivative** and is denoted $f''$. If the second derivative exists, we say that the original function is twice differentiable.
- This second derivative is again a function; hence, we can define its derivative (where it exists). The result is the **third derivative**, denoted $f'''$. If the third derivative exists, we say that the original function is three times differentiable.
- We can continue in this manner; from the fourth derivative on, we denote the derivatives by $f^{(4)}, f^{(5)}$, and so on.
  If the $n$th derivative exists, we say that the original function is $n$ times differentiable.





- Polynomials are functions that can be differentiated as many times as desired. The reason is that the first derivative of a polynomial of degree $n$ is a polynomial of degree $n-1$. Since the derivative is a polynomial as well, we can find its derivative, and so on. Eventually, the derivative will be equal to 0.
- We can write higher-order derivatives in Leibniz notation:
  The $n$th derivative of $f(x)$ is denoted by

$$\frac{d^n f}{dx^n}$$





## Example 6: (Online Homework HW13, # 4)

Find the first and second derivatives of the following function

$$f(x) = (5 - 3x^2)^4$$

## Example 7: (Online Homework HW13, # 16)

Find the first and second derivatives of the following function

$$y = \frac{1 - 4u}{1 + 3u}$$





## Velocity and Acceleration

The velocity of an object that moves on a straight line is the derivative of the objects position. The derivative of the velocity is the acceleration.

If $s(t)$ denotes the position of an object moving on a straight line, $v(t)$ its velocity, and $a(t)$ its acceleration, then the three quantities are related as follows:

$$v(t) = \frac{ds}{dt} \qquad \text{and} \qquad a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$





## Example 8: (Neuhauser, Problem # 87, p. 173)

Neglecting air resistance, the height $h$ (in meters) of an object thrown vertically from the ground with initial velocity $v_0$ is given by

$$h(t) = v_0 t - \frac{1}{2} g t^2$$

where $g = 9.81\text{m}/\text{s}^2$ is the earth's gravitational constant and $t$ is the time (in seconds) elapsed since the object was released.

**(a)** Find the velocity and the acceleration of the object.

**(b)** Find the time when the velocity is equal to 0. In which direction is the object traveling right before this time? in which direction right after this time?