WORKSHEET 12

1.

✻ $f(x) = x^3 - 5x^4 + 32$

instead of $x=3$ let's use a general value "a"

$$\frac{f(a+h)-f(a)}{h} = \frac{\left[(a+h)^3 - 5(a+h)^4 + 32\right] - \left[a^3 - 5a^4 + 32\right]}{h}$$

$$= \frac{\cancel{a^3} + 3a^2h + 3ah^2 + h^3 - 5(\cancel{a^4} + 4a^3h + 6a^2h^2 + 4ah^3 + h^3) + \cancel{32} - \cancel{a^3} + \cancel{5a^4} - \cancel{32}}{h}$$

$$= \frac{3a^2h + 3ah^2 + h^3 - 20a^3h - 30a^2h^2 - 20ah^3 - 5h^3}{h}$$

$$= 3a^2 + 3ah + h^2 - 20a^3 - 30a^2h - 20ah^2 - 5h^2$$

$$f'(a) = \lim_{h\to 0}\left[3a^2 + 3ah + h^2 - 20a^3 - 30a^2h - 20ah^2 - 5h^2\right]$$

$$= \boxed{3a^2 - 20a^3}$$

$$f'(3) = 3\cdot 3^2 - 20\,3^3$$
$$= -19\cdot 3^3 = \underline{\underline{-513}}$$

✻ $f(x) = 2 - \sqrt{7x^2 - 3}$

$$\frac{f(a+h)-f(a)}{h} = \frac{\left[2 - \sqrt{7(a+h)^2 - 3}\right] - \left[2 - \sqrt{7a^2 - 3}\right]}{h}$$

$$= \frac{-\sqrt{7(a+h)^2-3} + \sqrt{7a^2-3}}{h}$$

$$= \frac{\left(\sqrt{7a^2-3} - \sqrt{7(a+h)^2-3}\right)}{h} \frac{\left(\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}\right)}{\left(\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}\right)}$$

$$= \frac{(7a^2-3) - (7(a+h)^2-3)}{h\left(\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}\right)}$$

$$= \frac{7a^2 - 7(a+h)^2}{h\left(\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}\right)} = \frac{\cancel{7a^2} - \cancel{7a^2} - 14ah - 7h^2}{h\left(\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}\right)}$$

$$= \frac{\cancel{h}(-14a-7h)}{\cancel{h}\left(\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}\right)} = \frac{-14a-7h}{\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}}$$

$$f'(a) = \lim_{h \to 0} \frac{-14a-7h}{\sqrt{7a^2-3} + \sqrt{7(a+h)^2-3}} = \frac{-14a}{2\sqrt{7a^2-3}}$$

$$f'(3) = \frac{-14 \cdot 3}{2\sqrt{7 \cdot 3^2 - 3}} = \frac{-21}{\sqrt{60}} \approx -2.711$$

$\circledast$ $f(x) = \pi^4 - 17x^{-3/2}$

$$\frac{f(a+h)-f(a)}{h} = \frac{\left[\pi^4 - 17(a+h)^{-3/2}\right] - \left[\pi^4 - 17a^{-3/2}\right]}{h}$$

$$= \frac{17a^{-3/2} - 17(a+h)^{-3/2}}{h} =$$

$$= \frac{\dfrac{17}{a^{3/2}} - \dfrac{17}{(a+h)^{3/2}}}{h} =$$

$$= \frac{17(a+h)^{3/2} - 17a^{3/2}}{h \cdot a^{3/2} \cdot (a+h)^{3/2}} =$$

$$= \frac{17(a+h)\sqrt{a+h} - 17a\sqrt{a}}{h\,(a(a+h))^{3/2}} \cdot \frac{17(a+h)\sqrt{a+h} + 17a\sqrt{a}}{17(a+h)\sqrt{a+h} + 17a\sqrt{a}}$$

$$= \frac{17^{\cancel{2}}(a+h)^2(a+h) - 17^{\cancel{2}}a^2(a)}{h(a(a+h))^{3/2}\left[\cancel{17}(a+h)\sqrt{a+h} + \cancel{17}a\sqrt{a}\right]}$$

$$= \frac{17(a^2+2ah+h^2)(a+h) - 17a^3}{h(a(a+h))^{3/2}\left((a+h)\sqrt{a+h} + a\sqrt{a}\right)}$$

$$= \frac{\cancel{17a^3} + 17a^2h + 34a^2h + 34ah^2 + 17h^2a + 17h^3 \cancel{-17a^3}}{h(a(a+h))^{3/2}\left((a+h)\sqrt{a+h} + a\sqrt{a}\right)}$$

$$= \frac{51a^2\cancel{h} + 51a h^{\cancel{2}} + 17h^{\cancel{3}\,2}}{\cancel{h}\left(a(a+h)\right)^{3/2}\left((a+h)\sqrt{a+h} + a\sqrt{a}\right)}$$

$$= \frac{51a^2 + 51ah + 17h^2}{\left(a(a+h)\right)^{3/2}\left((a+h)\sqrt{a+h} + a\sqrt{a}\right)}$$

$$f'(a) = \lim_{h\to 0} \frac{51a^2 + \underbrace{51ah + 17h^2}_{\to 0}}{\left(a(a+h)\right)^{3/2}\left((a+h)\sqrt{a+h} + a\sqrt{a}\right)}$$

$$= \frac{51a^2}{(a^2)^{3/2}\cdot\left(a\sqrt{a} + a\sqrt{a}\right)} = \frac{51a^2}{a^3(2a\sqrt{a})}$$

$$= \frac{51}{2a^2\sqrt{a}} = \frac{51}{2a^{5/2}} = \boxed{\frac{51}{2}a^{-5/2}} \qquad f'(3) = \frac{51\,3^{-5/2}}{2}$$

$$\cong 1.6358$$

For the tangent line use:

$$\boxed{y - f(3) = f'(3)\cdot(x-3)}$$

$\boxed{2.}$ $$\lim_{x\to -6} \frac{f(x) - \frac{1}{2}}{x+6} = -\pi$$

The formula says that

$$f(-6) = 1/2 \qquad f'(-6) = -\pi$$

Hence at $P(-6, \frac{1}{2})$ the tg. line has equation

$$\boxed{y - \frac{1}{2} = -\pi\,(x+6)}$$

or $\underline{y = -\pi x - 6\pi + \frac{1}{2}}$

$\boxed{3.}$ $\displaystyle\lim_{h \to 0} \frac{4\sqrt[3]{8+h} - 8}{h}$

is $f'(a)$ where $\boxed{a = 8 \text{ and } f(x) = 4\sqrt[3]{x}}$

$\boxed{4.}$ f is differentiable at a point x_0 if $f'(x_0)$ exists. That is the tangent line is defined.

This is a <u>stronger</u> requirement.

differentiability implies continuity, but not viceversa. A function could be continuous at a point but not differentiable.

e.g. $f(x) = |x|$ is continuous at $x=0$ but not differentiable

$$f'(0) = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

does not exist.

$\boxed{5.}$ $f(x) = \begin{cases} ax^2 + bx + c & \text{if } x \le 0 \\ x^2 + 1 & \text{if } x > 0 \end{cases}$

We first need continuity at $x=0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

So $\lim_{x \to 0^-} (ax^2 + bx + c) = \lim_{x \to 0^+} (x^2 + 1)$

$$\boxed{c = 1}$$

We also want the derivative at $x=0$ to be defined

$$f'(x) = \begin{cases} 2ax + b & x<0 \\ 2x & x>0 \end{cases}$$

at $x=0$ we want

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$$

Thus $\lim_{x\to 0^-} (2ax+b) = \lim_{x\to 0^+} 2x$

$$\boxed{b=0}$$

Thus

$$f(x) = \begin{cases} ax^2+1 & x \le 0 \\ x^2+1 & x>0 \end{cases}$$

is a family of differentiable functions

"a" can be anything