WORKSHEET 23

1. (a) $\displaystyle\lim_{x\to 4} \frac{(4-x)^2}{16-x^2} = \frac{(4-4)^2}{16-4^2} = \frac{0}{0}$

using the substitution Thm

We can use l'Hôpital's Rule

$$= \lim_{x\to 4} \frac{2(4-x)(-1)}{-2x} = \lim_{x\to 4} \frac{4-x}{x} = \frac{4-4}{4} = \underline{\underline{0}}$$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{8-x+4x^2}}{5x+3} = \frac{\infty}{\infty}$

We can use l'Hôpital's Rule

$$= \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{8-x+4x^2}}(-1+8x)}{5}$$

$$= \lim_{x\to\infty} \frac{-1+8x}{10\sqrt{8-x+4x^2}}$$

<u>BUT</u> it is essentially the same as the original limit.

Thus we can look at the original limit

$$\lim_{x\to\infty} \frac{\sqrt{8-x+4x^2}\cdot \frac{1}{x}}{(5x+3)\cdot\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{8-x+4x^2}{x^2}}}{5+\frac{3}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{8}{x^2}-\frac{1}{x}+4}}{5+3/x}$$

$$= \frac{\sqrt{\lim_{x\to\infty}\left(\frac{8}{x^2}-\frac{1}{x}+4\right)}}{5+\lim_{x\to\infty}\frac{3}{x}} = \frac{\sqrt{4}}{5} = \frac{2}{5}$$

(c) $$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} = \frac{0}{0}$$

We can use l'Hôpital's Rule

$$= \lim_{x\to 0} \frac{e^x - 1}{2x} = \frac{0}{0}$$

We can use l'Hôpital's Rule again

$$= \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$

(d) $$\lim_{x\to 0^+} x^{3\sin(2x)} = 0^0$$

So we can rewrite

the limit as

$$\lim_{x \to 0^+} e^{\ln\left(x^{3\sin(2x)}\right)} =$$

$$= \lim_{x \to 0^+} e^{3\sin(2x) \cdot \ln(x)} =$$

$$= e^{\lim_{x \to 0^+} 3\sin(2x) \cdot \ln(x)} = e^{0 \cdot (-\infty)}$$

as the exponential is a continuous function

So we are left to consider

$$\lim_{x \to 0^+} 3\sin(2x) \cdot \ln x = \lim_{x \to 0^+} \frac{3 \cdot \ln(x)}{\frac{1}{\sin(2x)}} =$$

$$= \frac{-\infty}{\infty} = \text{ we can use l'Hôpital's Rule}$$

$$= \lim_{x \to 0^+} \frac{3 \cdot \frac{1}{x}}{\frac{-\cos(2x) \cdot 2}{\sin^2(2x)}} = \lim_{x \to 0^+} \frac{-3 \sin^2(2x)}{2 \quad x \cdot \cos(2x)}$$

$$= \lim_{x \to 0^+} \left(-\frac{3}{1}\right) \cdot \underbrace{\frac{\sin(2x)}{2x}}_{\to 1} \cdot \frac{\overbrace{\sin(2x)}^{\to 0}}{\underbrace{\cos(2x)}_{\to 1}} = \underline{\underline{0}}$$

Hence

$$e^{\lim_{x\to 0^+} 3\sin(2x)\cdot \ln x} = e^{0} = \underline{\underline{1}}$$

$\boxed{2.}$ (a) the equilibria of $x_{n+1} = \frac{1}{4}x_n^2 + x_n - \frac{1}{4}$

are obtained solving $\cancel{x} = \frac{1}{4}x^2 + \cancel{x} - \frac{1}{4}$

$\rightsquigarrow \quad \frac{1}{4}x^2 - \frac{1}{4} = 0 \quad \rightsquigarrow \quad x^2 = 1 \quad \rightsquigarrow \quad x = \pm 1$

So we have $\underline{\underline{\hat{x} = 1}}$ and $\underline{\underline{\hat{x} = -1}}$.

To determine the stability we need to use the stability criterion.

$f(x) = \text{updating function} = \frac{1}{4}x^2 + x - \frac{1}{4}$

$f'(x) = \frac{1}{4}\cdot 2x + 1 = \frac{1}{2}x + 1$

$$\begin{cases} f'(1) = \frac{1}{2}(1) + 1 = \frac{3}{2} \quad \text{so} \quad |f'(1)| = \frac{3}{2} > 1 \\ \text{So } \underline{\hat{x} = 1 \text{ is unstable}} \end{cases}$$

$f'(-1) = \frac{1}{2}(-1) + 1 = \frac{1}{2}$ so $|f'(-1)| = \frac{1}{2} < 1$

So $\underline{\hat{x} = -1}$ is $\underline{\text{locally stable}}$

(b) Consider $x_{t+1} = \dfrac{5x_t^2}{4+x_t^2} = f(x_t)$

with updating function $f(x) = \dfrac{5x^2}{4+x^2}$

The fixed (equilibria) points are:

$$x = \frac{5x^2}{4+x^2} \iff x(4+x^2) = 5x^2$$

$$\iff 9x + x^3 - 4x^2 = 0 \iff$$

$$x(x^2-5x+4) = 0 \iff x(x-4)(x-1) = 0$$

So $\boxed{\hat{x} = 0, 1, 4}$

Let' use the Stability criterion.

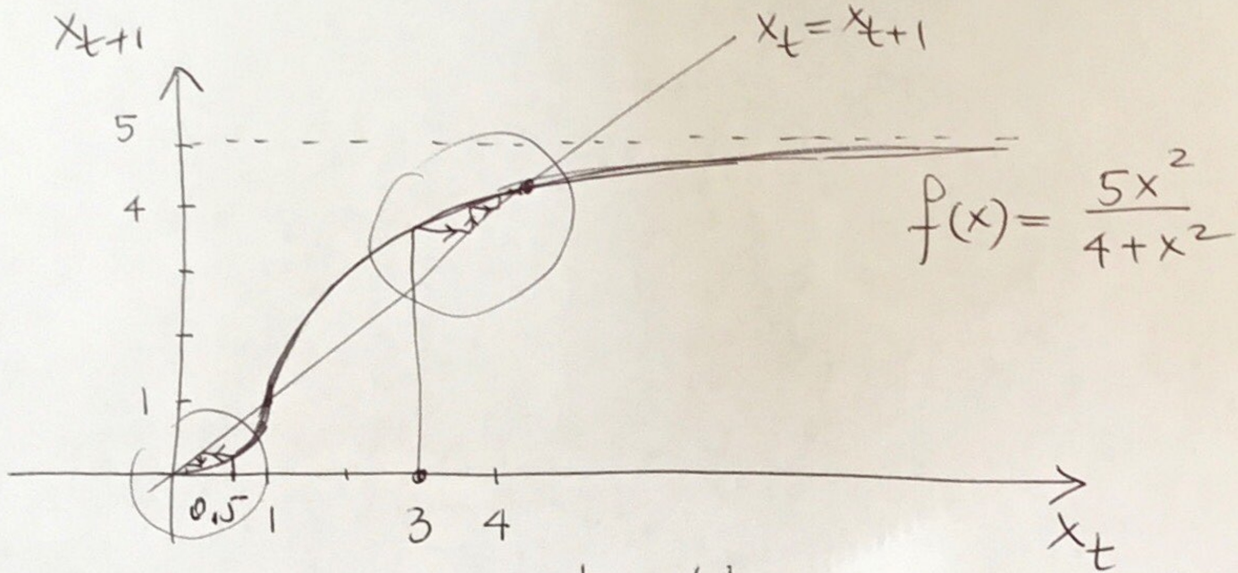
$$f'(x) = \frac{10x(4+x^2) - 5x^2(2x)}{(4+x^2)^2} = \frac{40x + \cancel{10x^3} - \cancel{10x^3}}{(4+x^2)^2}$$

$$= \frac{40x}{(4+x^2)^2}$$

$f'(0) = 0$ so $\hat{x} = 0$ is locally stable

$f'(1) = \dfrac{40}{25} > 1$ so $\hat{x} = 1$ is unstable

$f'(4) = \dfrac{160}{400} < 1$ so $\hat{x} = 4$ is locally stable



cobwebbing

(c)

$$N_{t+1} = \underbrace{R(N_t)} N_t \quad \text{when} \quad R(N) = r N^{1-\gamma}$$

So the recursive sequence is

$$N_{t+1} = \underbrace{r N_t^{1-\gamma}} \cdot N_t = \underbrace{r N_t^{2-\gamma}}$$

$f(x) = r \cdot x^{2-\gamma}$
is the updating function

To find the fixed points we need to solve $N = r N^{2-\gamma}$ so

$$N - r N^{2-\gamma} = 0 \iff N(1 - r N^{1-\gamma}) = 0$$

$$\iff \hat{N} = 0 \text{ or } 1 - r \hat{N}^{1-\gamma} = 0$$

$$\hat{N}^{1-\gamma} = \frac{1}{r} \Longrightarrow \hat{N} = \left(\frac{1}{r}\right)^{\frac{1}{1-\gamma}}$$

this is the non trivial equilibrium

To study its stability we need $f'(x)$

$$f(x) = r \cdot x^{2-\gamma}$$

$$f'(x) = r(2-\gamma) \cdot x^{(2-\gamma)-1} = r(2-\gamma) \cdot x^{1-\gamma}$$

Evaluate:

$$f'\left(\left(\frac{1}{r}\right)^{\frac{1}{1-\gamma}}\right) = r(2-\gamma) \cdot \left(\left(\frac{1}{r}\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}$$

$$= r(2-\gamma) \cdot \frac{1}{r} = 2-\gamma$$ for stability

We want $|2-\gamma| < 1$ so

$$-1 < 2-\gamma < 1$$

$\gamma < 3$ $\quad$ $1 < \gamma$

So $\boxed{1 < \gamma < 3}$ for stability