WORKSHEET #7

#1 Consider the recursive sequence

$$a_{n+1} = 13\left(a_n - 4a_n^2\right)$$

to find the fixed points (≡ i.e. possible candidates for $\lim_{n\to\infty} a_n$) we need to solve

$$a = 13\left(a - 4a^2\right)$$

$$52a^2 - 12a = 0$$

$$\Longleftrightarrow \quad 52a^2 - 13a + a = 0$$

$$\Longleftrightarrow \quad a(52a - 12) = 0 \Longleftrightarrow \boxed{\hat{a} = 0} \text{ OR } \boxed{\hat{a} = 12/52}$$

Suppose $a_0 = 1$

then $a_1 = 13\left(a_0 - 4a_0^2\right) = 13(1-4)$

$= -39$

$a_2 = 13\left(-39 - 4(-39)^2\right) = -79599$

$a_3 = 13\left(-79599 - 4(-79599)^2\right)$

it seems that

$$\lim_{n\to\infty} a_n = -\infty \quad \text{i.e.}$$

D.N.E.

$\boxed{2.}$ To find the fixed points of

$$a_{n+1} = \sqrt{7a_n}$$

we need to solve the equation

$$a = \sqrt{7a} \iff a^2 = 7a$$

$$\iff a^2 - 7a = 0 \iff a(a-7) = 0$$

thus $\boxed{\hat{a} = 0}$ OR $\boxed{\hat{a} = 7}$

Suppose now $a_0 = 1$. Then we have

$$a_1 = \sqrt{7a_0} = \sqrt{7 \cdot 1} = \sqrt{7} \approx 2.64575$$

$$a_2 = \sqrt{7a_1} = \sqrt{7\sqrt{7}} \approx 4.30352$$

$$a_3 = \sqrt{7a_2} = \sqrt{7\sqrt{7\sqrt{7}}} \approx 5.48859$$

$$a_4 = \sqrt{7a_3} = \sqrt{7\left(\sqrt{7\sqrt{7\sqrt{7}}}\right)} \approx 6.1984$$

$$a_5 = \sqrt{7a_4} \approx 6.587$$

it seems that

$$\lim_{n\to\infty} a_n = 7 \quad \text{starting with } a_0 = 1$$

$\boxed{\#3}$ The fixed points of the sequence

$a_{n+1} = \frac{5}{a_n}$ are given by $\boxed{a = \frac{5}{a}}$

so that $a^2 = 5 \iff a = \pm\sqrt{5}$

$\boxed{\hat{a} = \sqrt{5}}$ or $\boxed{\hat{a} = -\sqrt{5}}$

Suppose $a_0 = 1$ then $a_1 = \frac{5}{a_0} = \frac{5}{1} = 5$

$a_2 = \frac{5}{a_1} = \frac{5}{5} = 1$; $a_3 = \frac{5}{a_2} = \frac{5}{1} = 5$

$a_4 = \frac{5}{a_3} = \frac{5}{5} = 1$; $a_5 = \frac{5}{a_4} = \frac{5}{1} = 5$

So $a_0 = a_2 = a_4 = a_6 = \cdots \; a_{\text{even}} = 1$

$a_1 = a_3 = a_5 = a_7 = \cdots = a_{\text{odd}} = 5$

Values of the sequence keep alternating between 1 and 5 so $\underline{\underline{\lim_{n\to\infty} a_n = \text{D.N.E}}}$

$\boxed{\#4}$ $\lim_{x\to 4} \frac{(4-x)^2}{16-x^2}$

If we build a table of values

for $f(x) = \dfrac{(4-x)^2}{16-x^2}$ near 4 we obtain

x	3.9	3.99	4	4.01	4.1
$f(x)$	−0.0002308	−0.000002104		−0.000002062	−0.00018

it seems that $\displaystyle\lim_{x\to 4} \frac{(4-x)^2}{16-x^2} = 0$

Notice that we cannot substitute directly $x=4$ into the expression as we obtain

$$\frac{0}{0}$$

However we can manipulate the expression for $f(x)$ and obtain an equivalent one

$$\lim_{x\to 4} \frac{(4-x)^2}{16-x^2} = \lim_{x\to 4} \frac{(4-x)^{\cancel{2}}}{(\cancel{4-x})(4+x)} =$$

$$= \lim_{x\to 4} \frac{4-x}{4+x} = \frac{\lim_{x\to 4}(4-x)}{\lim_{x\to 4}(4+x)} = \frac{4-4}{4+4} = \frac{0}{8}$$

$$\boxed{= 0}$$