MA 137 — Calculus 1 with Life Science Applications
Derivatives of Trigonometric Functions
(Section 4.8)

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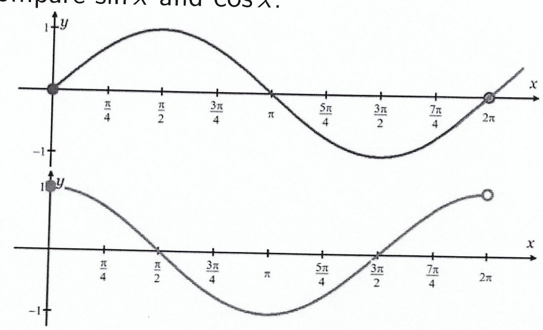
Cyclic phenomena are most easily modeled by sines and cosines:

- length of day;
- length of season;
- some population models (e.g. ideal predator-prey models).

We need to know how fast they change.
Let's compare $\sin x$ and $\cos x$:



The Derivative of Sine and Cosine

Theorem

The functions $\sin x$ *and* $\cos x$ *are differentiable for all* x*, and*

$$\frac{d}{dx}\sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx}\cos x = -\sin x$$

We need the trigonometric limits from Section 3.4 to compute the derivatives of the sine and cosine functions. Namely,

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0.$$

We also need the addition formulas for sine and cosine

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

Note that all angles are measured in radians.

Proof for Cosine

We use the formal definition of derivatives:

$$\begin{aligned}
\frac{d}{dx}\cos x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&\overset{\text{add. form.}}{=} \lim_{h\to 0}\frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\,(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0}\left[\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right] \\
&\overset{\text{laws}}{=} \cos x \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{\text{fund. lim.}}{=} \cos x \cdot 0 - \sin x \cdot 1 \\
&= -\sin x
\end{aligned}$$

Proof for Sine

We use the formal definition of derivatives:

$$
\begin{aligned}
\frac{d}{dx}\sin x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&\overset{\text{add. form.}}{=} \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\,(\cos h-1)+\cos x\sin h}{h} \\
&= \lim_{h\to 0}\left[\sin x\,\frac{\cos h-1}{h}+\cos x\,\frac{\sin h}{h}\right] \\
&\overset{\text{laws}}{=} \sin x\,\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\,\lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{\text{fund. lim.}}{=} \sin x\cdot 0+\cos x\cdot 1 \\
&= \cos x
\end{aligned}
$$

Derivatives of Remaining Trigonometric Functions

The derivatives of the other trigonometric functions can be found using the following identities and the quotient rule:

$$\tan x=\frac{\sin x}{\cos x} \qquad \cot x=\frac{\cos x}{\sin x}$$

$$\sec x=\frac{1}{\cos x} \qquad \csc x=\frac{1}{\sin x}$$

For example: $\frac{d}{dx}(\tan x)=\cdots=\sec^2 x=1+\tan^2 x.$

Example 1: (Online Homework HW15, # 3)

Find the equation of the tangent line to the curve $y=6x\cos x$ at the point $(\pi,-6\pi)$.

$y = 6x\cos x$

We want the equation of the tangent line at $P(\pi,-6\pi)$. Notice $y(\pi)=6\cdot\pi\cdot\underbrace{\cos(\pi)}_{-1} = -6\pi$ ✓

We need the derivative evaluated at $x=\pi$:

$$y' = 6\cdot 1\cdot\cos x + 6x\cdot(-\sin x) = 6\cos x - 6x\sin x$$

$$y'\big|_{x=\pi} = 6\cdot\underbrace{\cos\pi}_{-1} - 6\pi\underbrace{\sin(\pi)}_{=0} = \boxed{-6}$$

Hence $\boxed{y-(-6\pi) = -6(x-\pi)}$

OR $y = -6x + \cancel{6\pi} - \cancel{(6\pi)}$ $\therefore$ $\boxed{y=-6x}$

Example 2: (Online Homework HW15, # 4)

(a) Let $f(x) = \sin^3(x)$. Find $f'(x)$.

(b) Let $g(x) = \sin(x^3)$. Find $g'(x)$.

(a) $f(x) = \sin^3(x) = [\sin(x)]^3$

that's the meaning

So, using the power chain rule, we get

$$f'(x) = 3[\sin(x)]^{3-1} \cdot \cos(x)$$

$$= \boxed{3\sin^2(x) \cdot \cos(x)}$$

(b) $g(x) = \sin(x^3)$

Using the chain rule we get

$$g'(x) = \cos(x^3) \cdot \underbrace{3x^2}$$

$$= \boxed{3x^2\cos(x^3)}$$

Example 3: (Online Homework HW15, # 7)

Find the derivative of the following function:

$$f(x) = \frac{\cos(2x)}{6 - \sin(2x)}$$

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$$f'(x) = \frac{(\cos(2x))' \cdot (6 - \sin(2x)) - \cos(2x)\,(6 - \sin(2x))'}{[6 - \sin(2x)]^2}$$

$$= \frac{-\sin(2x) \cdot 2\,(6 - \sin(2x)) - \cos(2x) \cdot [-\cos(2x) \cdot 2]}{[6 - \sin(2x)]^2}$$

$$= \frac{-12\sin(2x) + \underbrace{2\sin^2(2x) + 2\cos^2(2x)}_{=2}}{[6 - \sin(2x)]^2}$$

$$= \boxed{\frac{2 - 12\sin(2x)}{[6 - \sin(2x)]^2}}$$

Example 4: (Online Homework HW15, # 8)

Find the derivative of the following function:

$$f(x) = (x^3 - \cos(6x^2))^5$$

$$f(x) = \left[x^3 - \cos(6x^2)\right]^5$$

We need to use the chain rule to find $f'(x)$:

$$f'(x) = 5\left[x^3 - \cos(6x^2)\right]^{5-1} \cdot \left(x^3 - \cos(6x^2)\right)'$$

$$= 5\left[x^3 - \cos(6x^2)\right]^4 \cdot \left(3x^2 - (-\sin(6x^2) \cdot 12x)\right)$$

$$= \boxed{5\left[x^3 - \cos(6x^2)\right]^4 \cdot \left(3x^2 + 12x\sin(6x^2)\right)}$$

Example 5:

Human heart goes through cycles of contraction and relaxation (called systoles). During cycles, blood pressure goes up and down repeatedly; as heart contracts, pressure rises, and as heart relaxes (for a split second), pressure drops.
Consider approximate function for blood pressure of a patient

$$P(t) = 100 + 20\cos\left(\frac{\pi t}{35}\right) \quad \text{mmHg}$$

where t is measured in minutes . Find and interpret $P'(t)$.

$$P(t) = 100 + 20\cos\left(\frac{\pi t}{35}\right)$$

$$P'(t) = 0 + 20 \cdot \left[-\sin\left(\frac{\pi t}{35}\right) \cdot \frac{\pi}{35}\right]$$

$$= -\frac{\cancel{20}^{\,4} \cdot \pi}{\cancel{35}_{\,7}} \sin\left(\frac{\pi t}{35}\right)$$

$$= \boxed{-\frac{4\pi}{7}\sin\left(\frac{\pi t}{35}\right)} \quad \text{mmHg/min}$$

$P'(t)$ is measured in mmHg/min; it is the change in blood pressure due to normal cycle of the heart.

Example 6: (Online Homework HW15, # 9)

During the human female menstrual cycle, the gonadotropin, FSH or follicle stimulating hormone, is released from the pituitary in a sinusoidal manner with a period of approximately 28 days.
Guyton's text on Medical Physiology shows that if we define day 0 ($t = 0$) as the beginning of menstruation, then FSH, $F(t)$, cycles with a high concentration of about 4.4 (relative units) around day 9 and a low concentration of about 1.2 around day 23.

a. Consider a model of the concentration FSH (in relative units) given by

$$F(t) = A + B\cos(\omega(t - \varphi)),$$

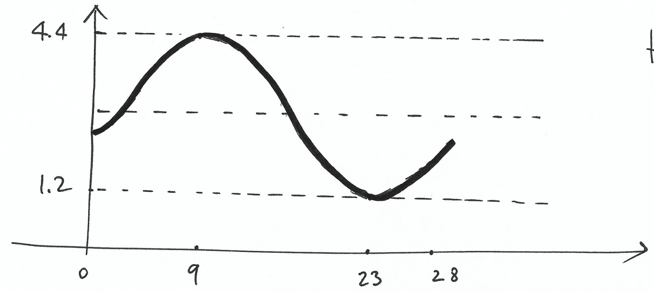
where A, B, ω, and φ (with $0 \leq \varphi \leq 28$) are constants and t is in days. Use the data above to find the four parameters.
If ovulation occurs around day 14, then what is the approximate concentration of FSH at that time?
You should sketch a graph of the concentration of FSH over one period.

b. Find the derivative of $F(t)$. Give its value at the time of ovulation ($t = 14$).

We need to find the correct values in

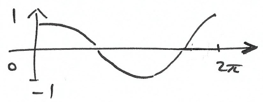
$$F(t) = A + B\cos(\omega(t - \varphi))$$

So that we have a complete cycle in 28 days; the high concentration is around day nine and has value 4.4; the low concentration is around day 23 and has value 1.2.



Here is how the graph should look like!

We need to shift, stretch and translate the graph of $\cos(t)$:



$A =$ gives the vertical shift of the x-axis

$B =$ gives half of the amplitude of the cycle.

A is the middle point between 4.4 and 1.2

$= 2.8 = \dfrac{4.4 + 1.2}{2}$

$B = \dfrac{4.4 - 1.2}{2}$ = half of the amplitude $= \underline{\underline{1.6}}$

the pick that occurs when $t = 0$ for $\cos(t)$ now occurs when $t = 9$. Thus $\underline{\underline{\varphi = 9}}$ gives the horizontal shift to the right

Finally we need to find the frequency "ω":

Notice that $F(t) = F(t + 28) = F(t + 56) = \ldots$

i.e. the values repeat every 28 days:

Thus: $F(t) = F(t+28)$ gives

$$A + B\cos(\omega(t - \varphi)) = A + B\cos(\omega(t + 28 - \varphi))$$

$$\Updownarrow$$

$$\cos(\omega(t - \varphi)) = \cos(\omega(t - \varphi) + \underline{\underline{\omega \cdot 28}})$$

Thus $\underline{\omega \cdot 28}$ must be $\underline{\underline{2\pi}}$

$$\therefore \ \omega = \frac{2\pi}{28} \simeq \underline{\underline{0.2244}}$$

Hence $\boxed{F(t) = 2.8 + 1.6\cos(0.2244(t-9))}$

(b) $F'(t) = 1.6\left[-\sin(0.2244(t-9)) \cdot (0.2244)\right]$

$= \boxed{-0.3590\sin(0.2244(t-9))}$ $\checkmark$

$F'(14) = -0.3590 \cdot \sin(1.122)$

$\cong -0.32345$

$F(14) = 2.8 + 1.6\cos(1.122) \cong 3.49421$