MA 137 — Calculus 1 with Life Science Applications
Derivatives of Exponential Functions
(Section 4.9)

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The Derivative of the Natural Exponential Function

Theorem

The function e^x is differentiable for all x, and $\dfrac{d}{dx}e^x = e^x$.

In particular, if $g(x)$ is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx}e^{g(x)} = e^{g(x)} \cdot g'(x).$$

We need to know the following limit to compute the derivative of the natural exponential function. Namely,

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

Although we cannot rigorously prove this result here, the table below should convince you of its validity

h	-0.1	-0.01	-0.001	$\cdots$	0.001	0.01	0.1
$\dfrac{e^h-1}{h}$	0.9516	0.9950	0.9995		1.0005	1.0050	1.0517

Proof

We use the formal definition of the derivative. In the final step, we will be able to write the term e^x in front of the limit because e^x does not depend on h.

$$\begin{aligned}
\frac{d}{dx}e^x &\overset{\text{def}}{=} \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\
&\overset{\text{exp. prop.}}{=} \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\
&= \lim_{h\to 0}\frac{e^x(e^h-1)}{h}\\
&\overset{\text{laws}}{=} e^x \lim_{h\to 0}\frac{e^h-1}{h}\\
&\overset{\text{fund. lim.}}{=} e^x\cdot 1\\
&= e^x
\end{aligned}$$

The Derivative of ANY Exponential Function

Theorem

The function a^x is differentiable for all x, and $\dfrac{d}{dx}a^x = a^x \cdot \ln a$.

In particular, if $g(x)$ is a differentiable function, it follows from the chain rule that

$$\frac{d}{dx}a^{g(x)} = a^{g(x)} \cdot \ln a \cdot g'(x).$$

We can prove the above result using the definition of the derivative and the limit

$$\lim_{h\to 0}\frac{a^h-1}{h} = \ln a,$$

in the same manner that we did for the natural exponential function.

Alternatively, we can use the following identity

$$a^x = e^{\ln a^x} = e^{x\ln a}$$

and the chain rule. Namely,

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot \ln a = a^x \cdot \ln a.$$

Example 1: (Nuehauser, Example # 1, p. 190)

Find the derivative of $f(x) = e^{-x^2/2}$.

$f(x) = e^{-x^2/2}$

We need to use the chain rule

$f'(x) = e^{-x^2/2} \cdot \frac{d}{dx}\left(-\frac{x^2}{2}\right)$

$= e^{-x^2/2} \cdot \left(-\frac{1}{2} \cdot 2x\right) = \boxed{-x e^{-x^2/2}}$

Example 2:

Find the derivative with respect to x of $g(x) = xe^{-x}$.

Evaluate $g'(x)$ at $x = 1$.

$g(x) = x e^{-x}$

$g'(x) = 1 \cdot e^{-x} + x \cdot \frac{d}{dx}\left(e^{-x}\right)$

product rule

$= e^{-x} + x\left[e^{-x}(-1)\right]$

chain rule

$= e^{-x} - x e^{-x}$

$\boxed{= e^{-x}(1-x)}$

$g'(1) = e^{-1} \cdot (1-1) = \underline{\underline{0}}$

Example 3: (Online Homework HW15, # 14)

The cutlassfish is a valuable resource in the marine fishing industry in China. A von Bertalanffy model is fit to data for one species of this fish giving the length of the fish, $L(t)$ (in mm), as a function of the age, a (in yr). An estimate of the length of this fish is

$$L(a) = 593 - 378e^{-0.166a}.$$

(a) Find the L-intercept.
Find an equation for the horizontal asymptote of $L(a)$.
Find the maximum possible length of this fish.

(b) Determine how long it takes for this fish to reach 90 percent of its maximum length.

(c) Differentiate $L(a)$ with respect to a.

(a) $L(a) = 593 - 378\, e^{-0.166a}$

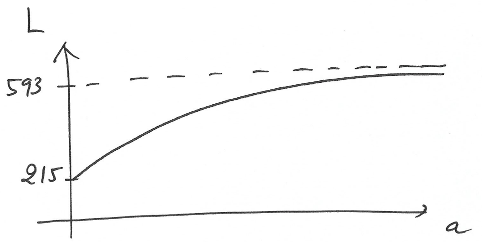
To find the L-intercept we set $a = 0$

$$L(0) = 593 - 378\, e^{-0.166a} = 593 - 378\, \underbrace{e^{0}}_{1} = 593 - 378 = \underline{215}$$

To get the equation of the horizontal asymptote we need to evaluate $\lim_{a\to\infty}\left(593 - 378\, e^{-0.166a}\right)$

$$= 593 - 378 \underbrace{\lim_{a\to\infty} e^{-0.166a}}_{0} = 593 - 0 = \underline{\underline{593}}$$

Hence the maximum possible length of the fish is (close to) 593



graph of $L(a)$

(b) We need to find the age such that

$$\underbrace{0.9 \cdot 593}_{\text{90\% of max. length}} = L(a) = 593 - 378\, e^{-0.166a}$$

$$\Longleftrightarrow \quad 378\, e^{-0.166a} = 593 - 533.7$$

$$\Longleftrightarrow \quad e^{-0.166a} = \frac{59.3}{378} \cong 0.15688$$

$$\Longleftrightarrow \quad -0.166a = \ln(0.15688)$$

$$\Longleftrightarrow \quad a = \frac{\ln(0.15688)}{-0.166} \cong \underline{\underline{11.1583}}$$

(c) $L(a) = 593 - 378\, e^{-0.166a}$

$$\frac{dL}{da} = L'(a) = 0 - 378 \cdot \underbrace{e^{-0.166a} \cdot (-0.166)}_{\text{chain rule}}$$

$$= 378 \cdot (0.166)\, e^{-0.166a}$$

$$= \underline{\underline{62.748\, e^{-0.166a}}}$$

notice that the derivative is always positive !!

Example 4: (Neuhauser, Example # 5, p. 191)

Exponential Growth: Show that the function $N(t) = N_0 e^{rt}$ satisfies the differential equation

$$\frac{dN}{dt} = rN(t) \qquad N(0) = N_0.$$

[N_0 is the population size at time $t = 0$ and r is called the growth rate.]

$N(t) = N_0 e^{rt}$

notice that at $t=0$ we get

$$N(0) = N_0 \underbrace{e^{r\cdot 0}}_{1} = N_0 \checkmark$$

Let's compute the derivative of $N(t) = N_0 e^{rt}$

$$\frac{dN}{dt} = N_0 \underbrace{e^{rt} \cdot (r)}_{\text{chain rule}}$$

Substitute in the D.E. $\frac{dN}{dt} = rN$

$$\underbrace{N_0 e^{rt}(r)} \stackrel{?}{=} r\left(N_0 e^{rt}\right)$$

the two sides are identical! ✓

Example 5: (Neuhauser, Example # 6, p. 192)

Radioactive Decay: Show that the function $W(t) = W_0 e^{-rt}$ satisfies the differential equation

$$\frac{dW}{dt} = -rW(t) \qquad W(0) = W_0.$$

[W_0 is the amount of material at time $t = 0$ and r is called the radioactive decay rate.]

$W(t) = W_0 e^{-rt}$

notice that at $t=0$

$$W(0) = W_0 \underbrace{e^{-r\cdot 0}}_{=1} = W_0 \checkmark$$

Let's compute the derivative of $W(t) = W_0 e^{-rt}$

$$\frac{dW}{dt} = W_0 \underbrace{e^{-rt} \cdot (-r)}_{\text{chain rule.}}$$

Substitute in the D.E. $\frac{dW}{dt} = -rW$

$$\underbrace{W_0 e^{-rt}(-r)} \stackrel{?}{=} -r\left(W_0 e^{-rt}\right)$$

the two sides are identical! ✓

Example 6: (Neuhauser, Problem # 63, p. 193)

(a) Find the derivative of the logistic growth curve (Example 4, Section 3.3, p. 123)

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right) e^{-rt}}$$

(b) Show that $N(t)$ satisfies the differential equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \qquad N(0) = N_0$$

(c) Plot the per capita rate of growth $\frac{1}{N}\frac{dN}{dt}$ as a function of N, and note that it decreases with increasing population size.

(a)

$$N(t) = \frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}} = K\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^{-1}$$

Let's compute the derivative using this form instead of the quotient rule:

$$\frac{dN}{dt} = N' = K\,(-1)\cdot\underbrace{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^{-2}\cdot\left(\frac{K}{N_0}-1\right)e^{-rt}(-r)}_{\text{chain rule}}$$

$$= \boxed{\frac{Kr\left(\frac{K}{N_0}-1\right)e^{-rt}}{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^2}}$$

Let us substitute the derivative and the function into the D.E $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$

$$\frac{dN}{dt} = \frac{Kr\left(\frac{K}{N_0}-1\right)e^{-rt}}{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^2} \overset{?}{=} rN\left(1-\frac{N}{K}\right)$$

$$\overset{?}{=} r\,\frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}\cdot\left(1-\frac{\cancel{K}}{\cancel{K}\left(1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right)}\right)$$

$$\overset{?}{=} r\,\frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}\cdot\frac{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]-1}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}$$

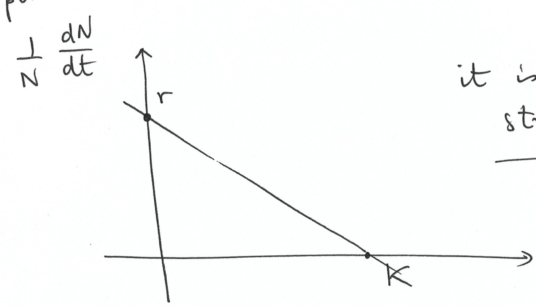
$$\overset{?}{=} \frac{rK\cdot\left(\frac{K}{N_0}-1\right)e^{-rt}}{\left[1+\left(\frac{K}{N_0}-1\right)e^{-rt}\right]^2}$$

Yes!!

(c) As we have discussed in a previous lecture

$$\frac{1}{N}\frac{dN}{dt} = r - \frac{r}{K}N$$

as a function of N has the following graph:



it is a decreasing straight line!