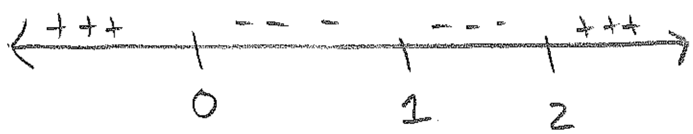
1 Let $f(x) = \dfrac{x^2}{x-1}$. Then $f'(x) = \dfrac{x^2-2x}{(x-1)^2}$ and $f''(x) = \dfrac{2}{(x-1)^3}$.

(a) Where are the critical points of f?

$f'(x) = 0 \Rightarrow x(x-2) = 0 \Rightarrow \boxed{x = 0, 2}$

$f'(x)$ not defined when $(x-1)^2 = 0 \Rightarrow \boxed{x = 1}$

(b) Find where f is increasing or decreasing.



x	numerator of f'	inc/dec
-1	$1+2$	inc
$1/2$	$\frac{1}{4} - \frac{1}{2}$	dec
$3/2$	$\frac{9}{4} - 3$	dec
3	$9-6$	inc

Only need to check the sign of the numerator, since $(x-1)^2 \geq 0$ for all x

f inc. on $(-\infty, 0) \cup (2, \infty)$
f dec. on $(0, 2)$

(c) Classify the critical points of f.

f' goes from $+$ to $-$ at $x = 0 \Rightarrow$ local max

f' goes from $-$ to $+$ at $x = 2 \Rightarrow$ local min

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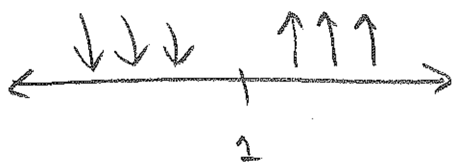
(d) Find where f is concave up or down.

$f''(x) \neq 0$

Not defined at $x=1$



$f''(2) = 2 \Rightarrow$ conc. up

$f''(0) = -2 \Rightarrow$ conc. down

f is conc. up on $(1, \infty)$

f is conc. down on $(-\infty, 1)$

(e) Identify any inflections points of f.

None. $x=1$ not an inflection point because f is not defined there.

(f) Find and classify any asymptotes of f.

$$\begin{array}{r|l} & x + 1 \\ \hline x-1 & x^2 + 0x + 0 \\ & -(x^2 - x) \\ \hline & x + 0 \\ & -(x-1) \\ \hline & 1 \end{array}$$

$$\Longrightarrow \frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}$$

$$\frac{1}{x-1} \to 0 \text{ as } x \to \infty$$

So $x+1$ is a slant/oblique asymptote

$\lim_{x \to 1^+} f(x) = \infty$, So $x=1$ is a vertical asymptote.

2 Blood pressure is the pressure exerted by circulating blood on walls of the arteries. Assume the pressure varies periodically according to the formula

$$p(t) = 90 + 15\sin(\pi t),$$

where t is the number of seconds since the beginning of a cardiac cycle.

(a) When is the blood pressure the highest for $0 \le t \le 2$? What is the maximum blood pressure?

$p(t)$ is continuous on $[0,2]$, so by EVT, it has a maximum on $[0,2]$ (and a minimum). This will occur at a critical point of $p(t)$, or an endpoint of $[0,2]$

$p'(t) = 15\pi\cos(\pi t)$

$p'(t) = 0 \Rightarrow \cos(\pi t) = 0 \Rightarrow \pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

Only $\frac{1}{2}$ and $\frac{3}{2}$ are in $[0,2]$

t	$p(t)$
0	90
1/2	105
3/2	75
2	90

$\leftarrow$ max

at $t = \frac{1}{2}$, blood pressure $= 105$

(b) When is the blood pressure lowest in the interval $0 \le t \le 2$ and what is the corresponding blood pressure?

From above work, we see that the minimum is at $t = 3/2$, and the blood pressure is 75.

3 There are roughly 20,000 UK undergraduates in Lexington. After classes finish for the year, students begin to leave town. Suppose that 6% of the students still in Lexington leave the area each day after the semester ends. Let N_t be the number of students in Lexington after t days.

(a) Find a recursion for N_t.

$N_0 = 20{,}000$

$$N_{t+1} = \underbrace{N_t}_{\text{current population}} - \underbrace{.06\,N_t}_{6\%\text{ loss}} = .94\,N_t$$

$$\boxed{\begin{array}{c} N_{t+1} = .94\,N_t \\ N_0 = 20{,}000 \end{array}}$$

(b) Find an explicit formula for N_t.

$N_1 = .94\,N_0$

$N_2 = .94\,N_1 = .94(.94\,N_0) = .94^2\,N_0$

$N_3 = .94\,N_2 = .94(.94^2\,N_0) = .94^3\,N_0$

$$\boxed{N_t = .94^t\,N_0 = 20{,}000 \times .94^t}$$

(c) How many students are still in Lexington two weeks after school ends?

Two weeks = 14 days

$N_{14} = 20{,}000 \times .94^{14} = 8410.46$

Need whole # of people $\Rightarrow \boxed{N_{14} = 8410}$

(a) Find a general formula for $\{a_n\}$ given a few terms of the sequence:

(i) $a_0 = \frac{1}{1}$ $\quad a_1 = -\frac{1}{3}$ $\quad a_2 = \frac{1}{5}$ $\quad a_3 = -\frac{1}{7}$ $\quad a_4 = \frac{1}{9}$ $\quad a_5 = -\frac{1}{11}$

Alternating positive/negative : $(-1)^n$

Odd numbers : $2n+1$

$$\Rightarrow \{a_n\} = \left\{\frac{(-1)^n}{2n+1}\right\}$$

(ii) $a_0 = 0$ $\quad a_1 = \frac{1}{2}$ $\quad a_2 = \frac{4}{4}$ $\quad a_3 = \frac{9}{8}$ $\quad a_4 = \frac{16}{16}$ $\quad a_5 = \frac{25}{32}$

Numerator : perfect squares $0, 1, 4, 9, \ldots$

Denominator : Powers of 2 $\quad 1, 2, 4, 8, \ldots$

$$\Rightarrow \{a_n\} = \left\{\frac{n^2}{2^n}\right\}$$

(b) Find the limit as $n \to \infty$ for the following sequences, if they exist.

(i) $a_n = \dfrac{n(3-2n^3)}{5n^4+n+1} = \dfrac{3n-2n^4}{5n^4+n+1} \cdot \dfrac{1/n^4}{1/n^4} = \dfrac{3/n^3 - 2}{5 + 1/n^3 + 1/n^4}$

$$\lim_{n\to\infty} \frac{3/n^3 - 2}{5 + 1/n^3 + 1/n^4} = \frac{0-2}{5+0+0} = \boxed{-\frac{2}{5}}$$

(ii) $a_n = \cos(n\pi)$

Write out a few terms :

$a_0 = \cos(0) = 1$

$a_1 = \cos(\pi) = -1$

$a_2 = \cos(2\pi) = 1$

$a_3 = \cos(3\pi) = -1$

$$\Rightarrow \{a_n\} = \{(-1)^n\}$$

Limit DNE. It oscillates between 1 and -1

5

(a) Find the fixed points of the recursion $a_{n+1} = \dfrac{a_n^2}{a_n^2 - 2}$.

Solve $a = f(a)$ where $f(x) = \dfrac{x^2}{x^2 - 2}$

$$a = \frac{a^2}{a-2} \rightarrow a^3 - 2a = a^2 \rightarrow a^3 - a^2 - 2a = 0$$

$$a[a^3 - a - 2] = 0$$

$$a(a-2)(a+1) = 0$$

$$\Rightarrow a^* = 0, -1, 2$$

(b) The fixed points of the recursion $x_{t+1} = rx_t - x_t^3$ are $x^* = 0$ and $x^* = \pm\sqrt{r}$. Classify their stability as a function of r. (Assume $r > 0$.)

Stability is determined by $|f'(x^*)|$. < 1 = stable, > 1 = unstable.

$f'(x) = r - 3x^2$

$|f'(0)| = |r| \rightarrow$ stable for $0 < r < 1$

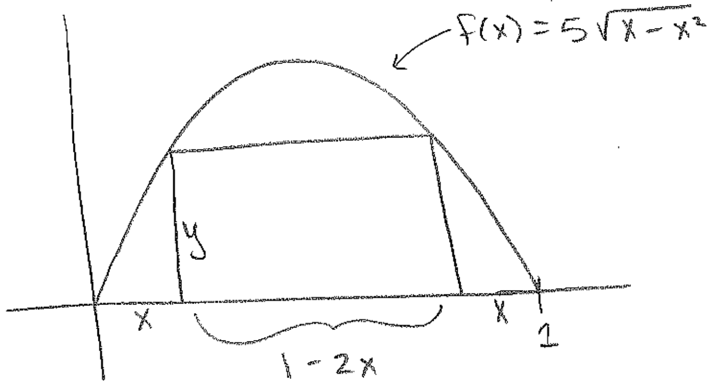
unstable for $r > 1$

$|f'(\pm\sqrt{r})| = |r - 3r| = |-2r| = 2r$

stable for $0 < r < \frac{1}{2}$

unstable for $r > \frac{1}{2}$

Bonus Find the largest rectangle whose base is on the x-axis and upper vertices are on the curve $f(x) = 5\sqrt{x - x^2}$.



Area = base × height

$A = (1-2x)\cdot y \qquad y = 5\sqrt{x-x^2}$

$A(x) = (1-2x)\cdot 5\sqrt{x-x^2} \qquad 0 < x < \frac{1}{2}$

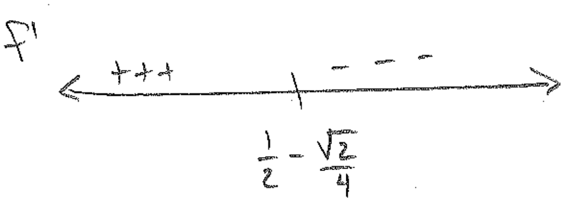
$A'(x) = (1-2x)\cdot 5 \cdot \frac{1}{2}(x-x^2)^{-1/2}\cdot(1-2x) + (-2)\cdot 5\sqrt{x-x^2} = 0$

$\Rightarrow \frac{5}{2}\frac{(1-2x)^2}{\sqrt{x-x^2}} = 10\sqrt{x-x^2} \Rightarrow (1-2x)^2 = 4(x-x^2)$

$\Rightarrow 4x^2 - 4x + 1 = 4x - 4x^2 \Rightarrow 8x^2 - 8x + 1 = 0$

$\Rightarrow x = \frac{8 \pm \sqrt{64 - 4\cdot 8}}{2\cdot 8} = \frac{1}{2} \pm \frac{\sqrt{2}}{4} \Rightarrow x = \frac{1}{2} - \frac{\sqrt{2}}{4}$

we need $x \in (0, \frac{1}{2})$



So $x = \frac{1}{2} - \frac{\sqrt{2}}{4}$ is a maximum.

Rectangle has area:

$A = \left[1 - 2\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)\right] 5\sqrt{\left(\frac{1}{2}-\frac{\sqrt{2}}{4}\right) - \left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right)^2}$