MAT 137, Spring 2014, Final Exam

Name ______________________ **Recitation** ______________________

ID ______________________

This exam is closed book and closed notes. Calculators are allowed, but cell phones, laptops, or any other electronic devices are prohibited.

Write clearly. Show your work where necessary. Answers without justification will receive little or no credit.

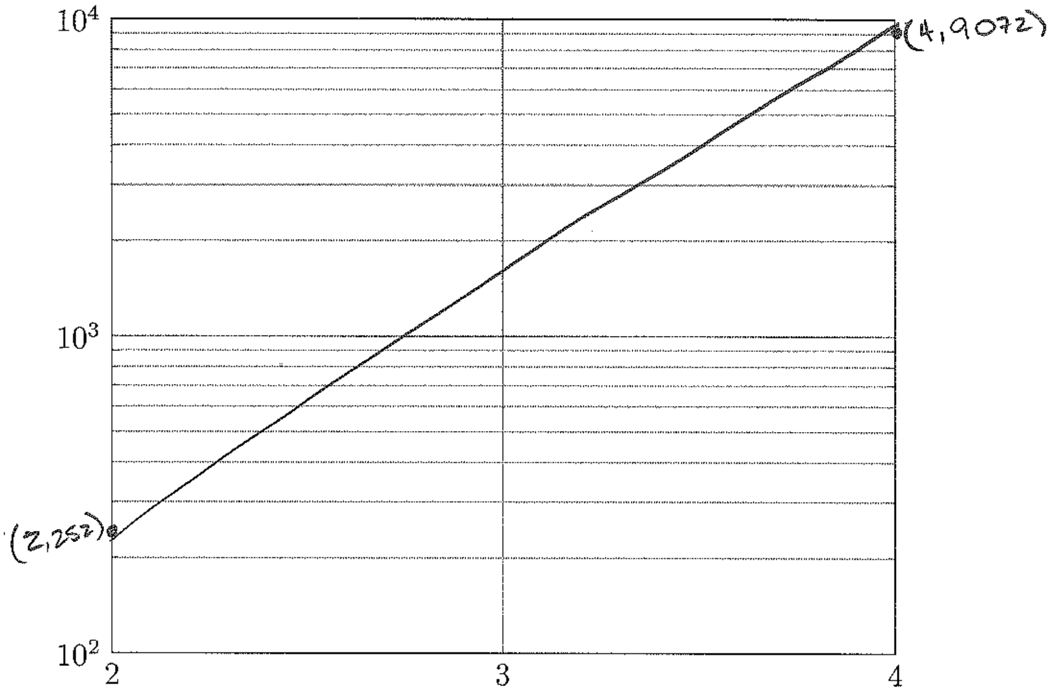
Question	Points	Possible
1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total		100

1

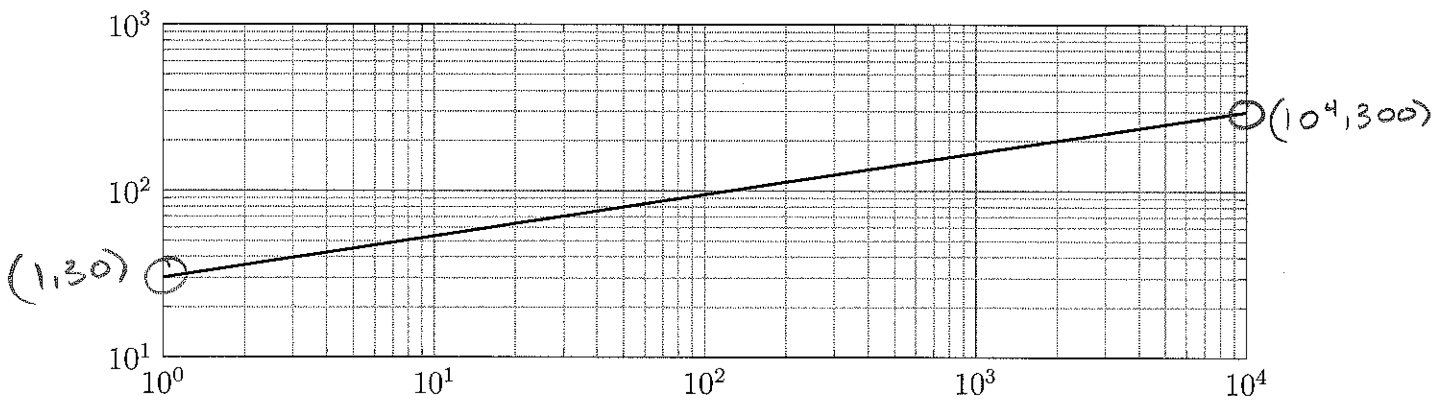
(a) Graph the curve $y = 7 \cdot 6^x$ on the semilog plot below. Clearly identify your points.

$7 \cdot 6^2 = 252$

$7 \cdot 6^4 = 9072$



(b) Determine the relationship between x and y from the following plot: (Use space on the back on the page if you need it.)



$y =$ $30 \cdot x^{1/4}$

Log-Log plot $\Rightarrow$ $y = bx^a$ (power Fcn)

$30 = b \cdot 1^a \rightarrow b = 30$

$300 = b \cdot (10^4)^a = 30 \cdot 10^{4a}$

$10 = 10^{4a} \rightarrow 4a = 1 \rightarrow a = \frac{1}{4}$

(a) Find the linear approximation to $f(x) = (x+1)^3$ at $x = 1$.

at $x=a$, $L(x) = f(a) + f'(a)(x-a)$

$f(1) = 2^3 = 8$

$f'(x) = 3(x+1)^2$

$f'(1) = 3 \cdot 2^2 = 12$

$L(x) = 8 + 12(x-1)$

Use this approximation to estimate $(2.03)^3$.

$(2.03)^3 = f(1.03) \approx L(1.03) = 8 + 12(1.03-1)$

$= 8 + 12 \cdot .03 = \boxed{8.36}$

(b) Use implicit differentiation to find $\frac{dy}{dx}$ if $\ln(y^2) = \sin(x^3 + y)$.

$\frac{1}{y^2} \cdot 2y \cdot y' = \cos(x^3+y) \cdot (3x^2 + y')$

$\frac{2}{y} y' - y' \cos(x^3+y) = 3x^2 \cos(x^3+y)$

$y' \left[\frac{2}{y} - \cos(x^3+y) \right] = 3x^2 \cos(x^3+y)$

$$y' = \frac{3x^2 \cos(x^3+y)}{\frac{2}{y} - \cos(x^3+y)}$$

3

(a) Find the value of c such that $f(x)$ is continuous everywhere.

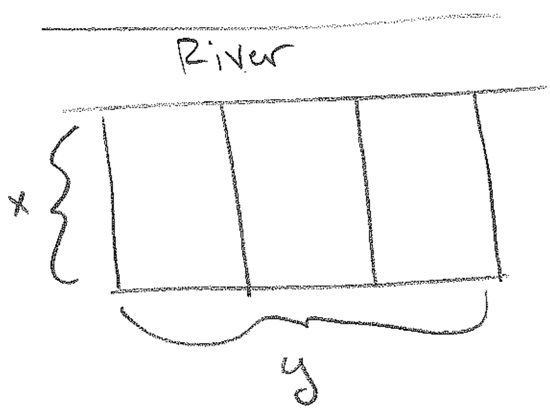
$$f(x) = \begin{cases} \frac{2}{x-1} + x^2 & x \leq 0 \\ \cos(x) + c & x > 0 \end{cases}$$

We need $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$

$$\Rightarrow \frac{2}{0-1} + 0^2 = \cos(0) + c$$

$$-2 = 1 + c \rightarrow \boxed{c = -3}$$

(b) A rectangular field is bounded on one side by a river and on the other three sides by a fence. Additionally, there are two interior fences, perpendicular to the river, that divide the field into three smaller enclosures. Find the largest field that can be enclosed in such a manner if the total of all fencing (interior and exterior) is 2400 ft.



Area $= x \cdot y$

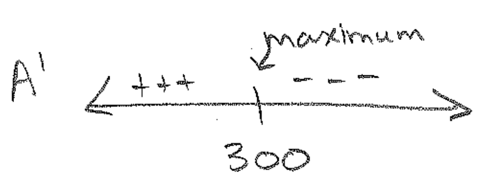
Total fence $= 2400 = 4x + y$

$\Rightarrow y = 2400 - 4x$

$\Rightarrow A(x) = x(2400 - 4x)$

$= 2400x - 4x^2$

$A'(x) = 2400 - 8x = 0 \Rightarrow 8x = 2400 \Rightarrow x = 300$



$y = 2400 - 4x$

$= 2400 - 1200$

$y = 1200$

Largest field is $300' \times 1200' = 360000 \text{ ft}^2$

4

(a) Find the derivative of $f(x) = \frac{(2x^3+4)^2}{\ln(x)-1}$

$$f'(x) = \frac{[\ln(x)-1]\cdot 2(2x^3+4)\cdot 6x^2 - (2x^3+4)^2\cdot \frac{1}{x}}{[\ln(x)-1]^2}$$

(b) Cardiac output (the volume of blood pumped by the heart per minute) can be calculated with the formula $C = \frac{Q}{D}$, where Q is the number of milliliters of CO_2 exhaled per minute and D is the difference between the CO_2 concentration in blood pumped to the lungs and blood returning from the lungs.

$\frac{dD}{dt} = -2$ Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. How is cardiac output changing at this moment?

$\frac{dQ}{dt} = 0$ (i.e. Q is a constant)

$$C = \frac{Q}{D}$$

$$\frac{dC}{dt} = -\frac{Q}{D^2}\cdot\frac{dD}{dt}$$

$$= -\frac{233}{41^2}\cdot(-2) = \boxed{\frac{466}{41^2}}$$

5 Let $f(x) = xe^{-x^2/2}$

(a) Find the intervals where $f(x)$ is increasing or decreasing.

$$f'(x) = xe^{-x^2/2} \cdot (-x) + e^{-x^2/2} = e^{-x^2/2}(1-x^2)$$

$$f'(x) = 0 \Rightarrow x = \pm 1 \qquad (e^{-x^2/2} \text{ always } > 0)$$

f'

$---$ | $+++$ | $---$

-1 $\quad$ 1

$f'(-2) < 0 \qquad f'(0) > 0 \qquad f'(2) < 0$

(b) Find the intervals where $f(x)$ is concave up or concave down.

$$f''(x) = e^{-x^2/2}(-2x) + e^{-x^2/2}(-x)(1-x^2)$$

$$= e^{-x^2/2}\left[-2x - x + x^3\right] = e^{-x^2/2}\left[x^3 - 3x\right]$$

$$f''(x) = 0 \Rightarrow x^3 - 3x = 0 \Rightarrow x(x^2-3) = 0 \qquad x = 0, \pm\sqrt{3}$$

f''

$---$ | $+++$ | $---$ | $+++$

$-\sqrt{3}$ $\quad$ 0 $\quad$ $\sqrt{3}$

(c) Identify any extrema or inflection points.

From (a), we have a local min at $x = -1$, and a local max at $x = 1$

From (b), we have inflection points at $x = 0, \pm\sqrt{3}$

(d) Show that f has a horizontal asymptote by showing $\lim_{x\to\infty} f(x)$ exists. Where is the asymptote?

$$\lim_{x\to\infty} xe^{-x^2/2} = \lim_{x\to\infty} \frac{x}{e^{x^2/2}} = \frac{\infty}{\infty} \qquad \text{Apply L'Hopital: } = \lim_{x\to\infty} \frac{1}{xe^{x^2/2}} = 0$$

Horizontal asymptote at $y = 0$

6

(a) Find and classify the fixed points of the recursion $a_n = \frac{1}{3} - \frac{4}{3}a_n^2$.

Fixed points satisfy $a = f(a)$.

$\Rightarrow \quad a = \frac{1}{3} - \frac{4}{3}a^2 \quad \rightarrow \quad 3a = 1 - 4a^2 \quad \rightarrow \quad 4a^2 + 3a - 1 = 0$

$(4a-1)(a+1) = 0$

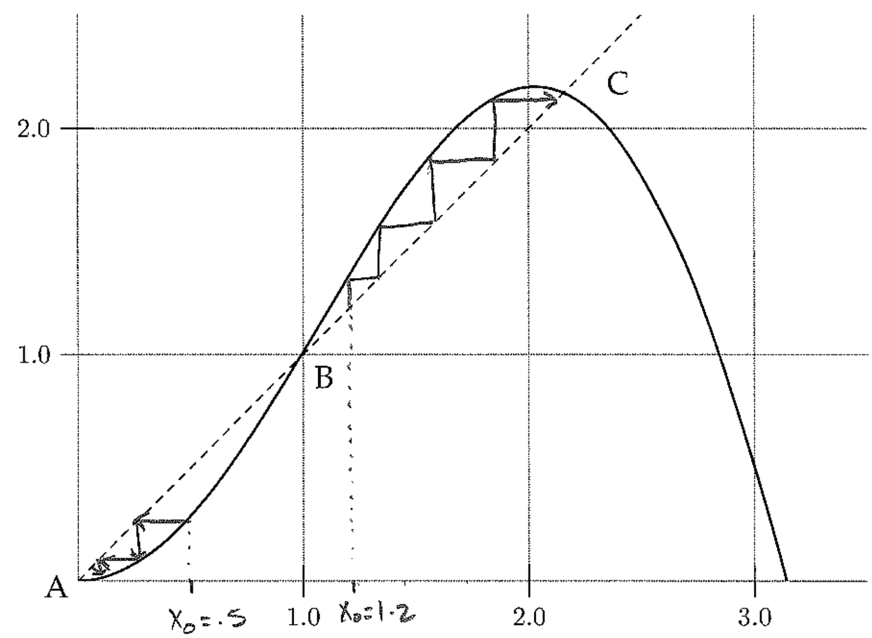
$a = \frac{1}{4}, -1$

Stability: $f'(x) = -\frac{8}{3}x$

$|f'(\frac{1}{4})| = |-\frac{8}{3} \cdot \frac{1}{4}| = \frac{2}{3} < 1 \Rightarrow$ stable

$|f'(-1)| = |-\frac{8}{3} \cdot (-1)| = \frac{8}{3} > 1 \Rightarrow$ unstable

(b) Let $x_{n+1} = f(x_n)$ be an unknown recursion, with the graph of $f(x_n)$ shown below. Use cobwebbing with the initial conditions $x_0 = .5$ and $x_0 = 1.2$ to determine the stability of the fixed points A, B, and C.



Identify (circle) the stability of each fixed point based on your cobwebbing:

A is stable/unstable B is stable/unstable C is stable/unstable

7 Compute the following limits or explain why they don't exist.

(a) $\lim_{z\to 2} \frac{z^3 - 5z + 2}{z^2 - 1}$ $= \frac{2^3 - 5(2) + 2}{2^2 - 1} = \frac{0}{3} = \boxed{0}$

(b) $\lim_{x\to\infty} \frac{\sin(x)}{x}$ Observe that $-\frac{1}{x} \leq \frac{\sin(x)}{x} \leq \frac{1}{x}$

And $\lim_{x\to\infty} -\frac{1}{x} = \lim_{x\to\infty} \frac{1}{x} = 0$

So by the Squeeze / Sandwich Theorem,

$\lim_{x\to\infty} \frac{\sin(x)}{x} = \boxed{0}$

(c) $\lim_{x\to\infty} x^{1/x}$ of the form ∞^0, so we use logarithms to transform

$= \exp\left[\lim_{x\to\infty} \frac{\ln(x)}{x}\right]$ $\lim_{x\to\infty} \frac{\ln(x)}{x} = \frac{\infty}{\infty} \Rightarrow$ Apply L'H.

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \frac{0}{1} = 0$

$= \exp[0]$

$\boxed{= 1}$

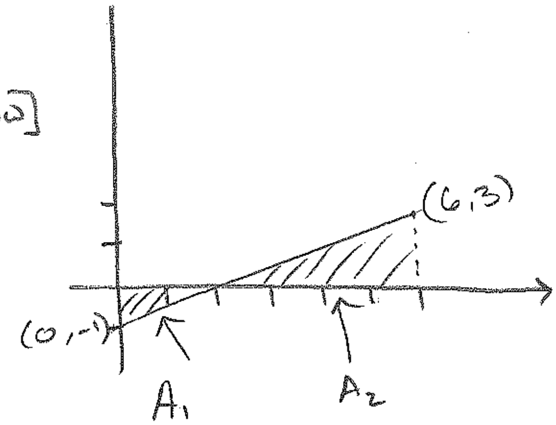
8

(a) Evaluate the definite integral $\int_0^6 \left(\frac{1}{2}x - 1\right)\,dx$ using only geometry. (Draw a picture first.)

$$\int_0^6 \left(\frac{1}{2}x-1\right)dx = [\text{Area Above x-axis}] - [\text{Area below}]$$

$$= A_2 - A_1$$

$$= 4 - 1 = \boxed{3}$$



$$A_1 = \frac{1}{2}\text{base} \times \text{height}$$

$$= \frac{1}{2}\cdot 2 \cdot 1 = 1$$

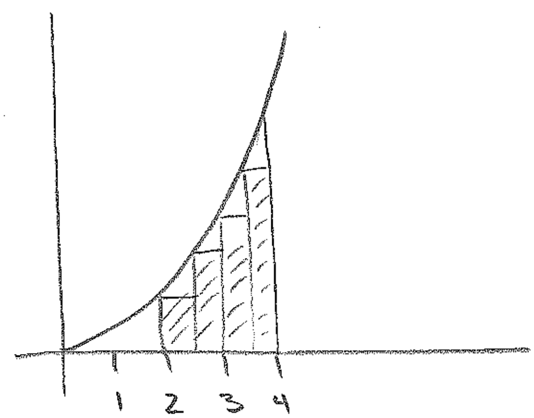
$$A_2 = \frac{1}{2}\text{base} \times \text{height}$$

$$= \frac{1}{2}\cdot 4 \cdot 2 = 4$$

(b) Estimate the area under the curve $y = x^2$ from $x = 2$ to $x = 4$ using 4 subintervals of equal width and left-hand endpoints.

$$\text{Width} = \Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2}$$

Intervals are: $[2,2.5], [2.5,3],$
$[3,3.5], [3.5,4]$



Left endpoints: $\{2, 2.5, 3, 3.5\}$

$$\text{Area} = \Delta x[f(2) + f(2.5) + f(3) + f(3.5)]$$

$$= \frac{1}{2}\left[2^2 + 2.5^2 + 3^2 + 3.5^2\right] = 15.75$$

9

(a) Evaluate $\int \left(x^3 - 2\cos(3x) + \frac{2}{x}\right) dx$

$= \frac{x^4}{4} - \frac{2}{3}\sin(3x) + 2\ln|x| + C$

(b) Evaluate $\int_1^3 e^x(1 - e^{-x})dx$

$= \int_1^3 (e^x - 1)\,dx = e^x - x \Big|_1^3 = \boxed{(e^3 - 3) - (e - 1)}$

(c) Solve the initial-value problem $\frac{dW}{dt} = e^{-2t}$ with $W(0) = 2$.

$W(t) = \frac{-1}{2}e^{-2t} + C$

$W(0) = 2 = -\frac{1}{2} + C \Rightarrow C = \frac{5}{2}$

$W(t) = \frac{-1}{2}e^{-2t} + \frac{5}{2}$

10

(a) Find the derivative of $F(x) = \int_x^3 \ln(2t)\sin(t)dt$.

$$F(x) = -\int_3^x \ln(2t)\sin(t)dt$$

By FTC 1, $F'(x) = -\ln(2x)\sin(x)$

(b) Use the Fundamental Theorem of Calculus to evaluate

$$\frac{d}{dx}\int_x^{2x^3-x} e^t\sqrt{t}dt$$

By FTC 1 / Leibniz's Formula:

$$= e^{2x^3-x}\sqrt{2x^3-x}\cdot(6x^2-1) - e^x\sqrt{x}$$

(c) Suppose f is continuous on $[1,4]$. Given that

$$\int_1^3 f(x)dx = 7 \qquad \int_2^4 f(x)dx = -3 \qquad \int_3^4 f(x)dx = 16,$$

find $\int_1^2 f(x)dx$.

$$\int_1^2 f(x)dx = \int_1^3 f(x)dx + \int_3^4 f(x)dx - \int_2^4 f(x)dx$$

$$= 7 + 16 - (-3) = \boxed{26}$$