

# MA 137: Calculus I for the Life Sciences

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## Section 4.7: Derivatives of Logarithmic Functions

### Theorem (The Derivative of the Natural Exponential Function)

*The function  $\ln x$  is differentiable for all  $x > 0$ , and*

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

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*In particular, if  $f(x)$  is a differentiable function, it follows from the chain rule that*

$$\frac{d}{dx} \ln f(x) = \frac{1}{f(x)} \cdot f'(x).$$

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We can use the derivative of  $e^x$  and the relationship between the exponential and the natural logarithmic functions to find the derivative of the function  $\ln x$ . Namely, we start by taking the derivative with respect to  $x$  of both sides of  $e^{\ln x} = x$ . We obtain

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$$e^{\ln x} \frac{d}{dx} \ln x = 1 \quad \text{then} \quad \frac{d}{dx} \ln x = \frac{1}{x}.$$

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### Alternative Proof.

We use the formal definition of the derivative and the following limit

$$e^x = \lim_{u \rightarrow \infty} \left(1 + \frac{x}{u}\right)^u$$

Then

$$\frac{d}{dx} \ln x = \lim_{h \rightarrow 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \ln \left( \frac{x+h}{x} \right) = \lim_{h \rightarrow 0} \frac{1}{x} \frac{x}{h} \ln \left( 1 + \frac{h}{x} \right).$$

Setting  $u = x/h$  we have,

$$\frac{d}{dx} \ln x = \frac{1}{x} \lim_{u \rightarrow \infty} \ln \left( 1 + \frac{1}{u} \right)^u = \frac{1}{x} \ln \left( \lim_{u \rightarrow \infty} \left( 1 + \frac{1}{u} \right)^u \right) = \frac{1}{x} \ln e = \frac{1}{x}.$$

## Section 4.7: Derivatives of Logarithmic Functions

### Theorem (The Derivative of Any Logarithmic Function)

*The function  $\log_a x$  is differentiable for all  $x > 0$ , and*

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From the base change formula for logarithms we have that

$$\log_a x = \frac{\ln x}{\ln a}$$

Thus it is enough to find the derivative of  $\ln x$ .

## Section 4.7: Derivatives of Logarithmic Functions

Example (Nuehauser, Problems # 28, 34, 52, p. 192)

Find the derivative of

1  $f(x) = \ln(1 - x^2).$

2  $f(x) = [\ln(1 - x^2)]^3.$

3  $f(s) = \ln(\ln s).$

## Section 4.7: Derivatives of Logarithmic Functions

Example (Nuehauser, Problems # 56, p. 193)

Find the derivative of

$$y = \log(3x^2 - x + 2)$$

## Section 4.7: Derivatives of Logarithmic Functions

### Example (Nuehauser, Problems # 62, p. 193)

Assume that  $f(x)$  is differentiable with respect to  $x$ . Show that

$$\frac{d}{dx} \ln \left[ \frac{f(x)}{x} \right] = \frac{f'(x)}{f(x)} - \frac{1}{x}$$

## Section 4.7: Logarithmic Differentiation

In 1695, Leibniz introduced logarithmic differentiation, following Johann Bernoulli's suggestion to find derivatives of functions of the form

$$y = [f(x)]^x$$

Bernoulli generalized this method and published his results two years later.

**The basic idea is to take logarithms on both sides and then to use implicit differentiation.**

## Section 4.7: Derivatives of Logarithmic Functions

Example (Nuehauser, Problems # 10, p. 190)

Find  $\frac{dy}{dx}$  when  $y = x^x$ .

What about  $\frac{d}{dx} [(2x)^{2x}]$ ?

## Section 4.7: Derivatives of Logarithmic Functions

### Example (Nuehauser, Problems # 66, 73, 74, p. 193)

Use logarithmic differentiation to find the first derivative of the functions

1  $y = (\ln x)^{3x}$

2  $y = x^{\cos x}$

3  $y = (\cos x)^x$

## Section 4.7: Derivatives of Logarithmic Functions

Example (Nuehauser, Problems # 75, p. 193)

Use logarithmic differentiation to find the first derivative of the function

$$y = \frac{e^{2x}(9x - 2)^3}{\sqrt[4]{(x^2 + 1)(3x^3 - 7)}}$$

# Section 4.7: Logarithmic Differentiation

## Theorem (Power Rule (General Form))

Let  $f(x) = x^r$ , where  $r$  is any real number. Then

$$\frac{d}{dx} x^r = r x^{r-1}$$

## Proof.

We set  $y = x^r$  and use logarithmic differentiation to obtain

$$\frac{d}{dx} \ln(y) = \frac{d}{dx} \ln(x)^r = \frac{d}{dx} r \ln(x) = r \frac{1}{x}$$

Then

$$\frac{1}{y} \frac{dy}{dx} = \frac{r}{x}$$

Solving for  $dy/dx$  yields

$$\frac{dy}{dx} = \frac{r}{x} y = \frac{r}{x} x^r = r x^{r-1}.$$

