MA 138 – Calculus 2 with Life Science Applications
Integration by Parts
(Section 7.2)

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About Example 4 from the previous lecture

Last time we integrated $\displaystyle\int \frac{3}{3+e^x}\,dx$ by using the substitution $u = 3+e^x$.

This lead to $du = e^x\,dx = (u-3)\,dx$. Thus

$$\int \frac{3}{3+e^x}\,dx \quad \leadsto \quad \int \frac{3}{u}\cdot\frac{du}{u-3} = \int \frac{3}{u(u-3)}\,du.$$

A natural question to ask is:

"Why should I care about integrals of this form?"

Next, I will give you a good reason.

We will study more systematically integrals of this form in Section 7.3.

The Logistic Growth Model

In Sections 3.3 and 4.1 we have introduced the logistic growth model. In this growth model it is assumed that the population size $N(t)$ at time t satisfies the initial value problem

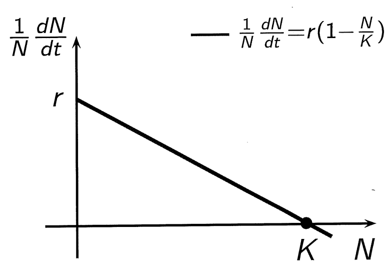
$$\frac{dN}{dt} = r\,N\left(1-\frac{N}{K}\right) \qquad N(0) = N_0,$$

where r (=growth rate) and K (=carrying capacity) are positive constants.

Rewriting this differential equation as

$$\frac{1}{N}\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)$$

says that the per capita growth rate in the logistic equation is a linearly decreasing function of population size.



In Chapter 8 we will see that in order to solve the logistic differential equation we first separate the variables to obtain

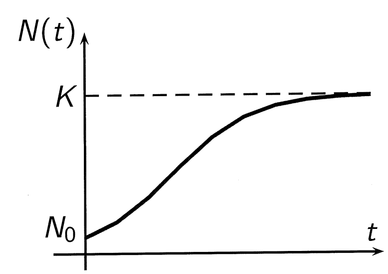
$$\frac{1}{N(1-N/K)}\,dN = r\,dt.$$

Then we integrate both sides with respect to N and t

$$\int \frac{K}{N(N-K)}\,dN = \int -r\,dt.$$

After several calculations we obtain that the solution of the IVP is

$$N(t) = \frac{K}{1+(K/N_0-1)e^{-rt}}.$$



Section 7.2: Integration by Parts

Integration by parts is the product rule in integral form.

Let $f = f(x)$ and $g = g(x)$ be differentiable functions. Then, differentiating the product fg with respect to x yields

$$(fg)' = f'g + fg'$$

or, after rearranging,

$$fg' = (fg)' - f'g.$$

Integrating both sides with respect to x, we find that

$$\int fg'\,dx = \int (fg)'\,dx - \int f'g\,dx.$$

Since fg is an antiderivative of $(fg)'$, it follows that

$$\int (fg)'\,dx = fg + C.$$

Therefore

$$\int fg'\,dx = fg - \int f'g\,dx.$$

(Note that the constant C can be absorbed into the indefinite integral on the right-hand side.) Because $f' = df/dx$ and $g' = dg/dx$, we can also write the preceding equation in the short form

$$\int f\,dg = fg - \int g\,df.$$

We summarize this discussion, by stating the following **general rule**:

Rule for Integration by Parts

If $f(x)$ and $g(x)$ are differentiable functions, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx;$$

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 1 (Problem #61, Section 7.2, page 373)

Evaluate the indefinite integral: $\int \ln x\,dx$.

Integration by parts says that

$$\int f g'\,dx = fg - \int f' g\,dx$$

In our case:

$$\int \ln x\,dx = \int \underbrace{(\ln x)}_{f} \cdot \underbrace{1}_{g'} \cdot dx$$

Hence:

$$\int \ln x\,dx = \underbrace{x}_{g} \cdot \underbrace{\ln x}_{f} - \int \underbrace{\frac{1}{x}}_{f'} \cdot x\,dx$$

$$= \int \ln x\,dx = x \cdot \ln x - \int 1 \cdot dx = \underline{x \cdot \ln x - x + C}$$

Example 2 (Online Homework # 8)

If $g(1) = -5$, $g(5) = 2$ and $\int_1^5 g(x)\,dx = -10$, evaluate

$$\int_1^5 x\,g'(x)\,dx.$$

$$\int_1^5 x\,g'(x)\,dx = ?$$

$g(1) = -5$

$g(5) = 2$

$\int_1^5 g(x)\,dx = -10$

integration by parts:

$$\int_1^5 x\,g'(x)\,dx = x\cdot g(x)\Big]_1^5 - \int_1^5 \underbrace{1}_{\text{derivative of } f(x)=x}\cdot g(x)\,dx =$$

$$= 5\cdot g(5) - 1\cdot g(1) - \underbrace{\int_1^5 g(x)\,dx}_{-10} =$$

$$= 5\cdot 2 - 1\cdot(-5) - (-10) = 10 + 5 + 10 = \underline{25}$$

Example 3 (Online Homework # 2)

Evaluate the indefinite integral: $\int e^{4x}\sin(6x)\,dx.$

$$\int \underbrace{e^{4x}}_{g'}\cdot\underbrace{\sin(6x)}_{f}\,dx = \underbrace{\left(\tfrac{1}{4}e^{4x}\right)}_{g}\cdot\underbrace{\sin(6x)}_{f} - \int \underbrace{\tfrac{1}{4}e^{4x}}_{g}\,\underbrace{\cos(6x)\cdot 6}_{f'}\,dx$$

$$\therefore \int e^{4x}\sin(6x)\,dx = \tfrac{1}{4}e^{4x}\cdot\sin(6x) - \tfrac{3}{2}\int \underbrace{e^{4x}}_{g'}\cdot\underbrace{\cos(6x)}_{f}\,dx$$

now what? Let's try integration by parts again

$$= \tfrac{1}{4}e^{4x}\sin(6x) - \tfrac{3}{2}\left[\tfrac{1}{4}e^{4x}\cos(6x) - \int\left(\tfrac{1}{4}e^{4x}\right)\cdot\left(-\sin(6x)\cdot 6\right)dx\right.$$

$$= \tfrac{1}{4}e^{4x}\sin(6x) - \tfrac{3}{8}e^{4x}\cos(6x) - \tfrac{9}{4}\int e^{4x}\sin(6x)\,dx$$

Now move the integral on the right-hand side of the inequality to the left-hand side.

We can add those 2 integrals

$$\underbrace{\left(1+\frac{9}{4}\right)}_{\frac{13}{4}} \int e^{4x} \cdot \sin(6x)\,dx = \frac{1}{4}e^{4}\sin(6x) - \frac{3}{8}e^{4x}\cos(6x) + C$$

$\therefore$

$$\int e^{4x} \cdot \sin(6x)\,dx = \frac{4}{13}\left[\frac{1}{4}e^{4}\sin(6x) - \frac{3}{8}e^{4x}\cos(6x)\right] + \tilde{C}$$

$$= \frac{1}{13}e^{4}\sin(6x) - \frac{3}{26}e^{4x}\cos(6x) + \tilde{C}$$

Example 4 (Online Homework # 3)

Evaluate the indefinite integral: $\int x^9 \cos(x^5)\,dx.$

(**Hint:** First make a substitution and then use integration by parts to evaluate the integral.)

$\int x^9 \cdot \cos(x^5)\,dx$ = note that we can rewrite it as follows

$$= \int \underbrace{x^4 \cdot x^5}_{=x^9} \cdot \cos(x^5)\,dx$$

So, set $u = x^5$ and observe that $\frac{du}{dx} = 5x^4$

so $\frac{1}{5}du = x^4\,dx$. Thus

$$\int x^9 \cos(x^5)\,dx = \int \underbrace{\frac{1}{5} \cdot u}_{f}\ \underbrace{\cos(u)}_{g'}\,du =$$

use now integration by parts!

$$= \underbrace{\frac{1}{5}u}_{f} \cdot \underbrace{\sin(u)}_{g} - \int \underbrace{\frac{1}{5}}_{f'}\ \underbrace{\sin(u)}_{g}\,du = \frac{1}{5}u\sin(u) + \frac{1}{5}\cos(u) + C$$

$\therefore$

$$\int x^9 \cos(x^5)\,dx \underset{u=x^5}{=} \int \frac{1}{5}u\cos(u)\,du$$

$$\underset{\text{by parts}}{=} \frac{1}{5}u\sin(u) + \frac{1}{5}\cos(u) + C$$

$$= \frac{1}{5}x^5\sin(x^5) + \frac{1}{5}\cos(x^5) + C$$

Example 5 (Problem #31, Section 7.2, page 372)

Evaluate the indefinite integrals:

$$\int \cos^2 x\, dx \qquad \int \cos^3 x\, dx.$$

$\boxed{\int \cos^3(x)\,dx}$ is easy! In fact $\cos^3(x) = \cos(x)(\cos^2 x)$

$= \cos(x)\cdot[1-\sin^2(x)]$

$$= \int \cos(x)\cdot[1-\sin^2(x)]\,dx \underset{u=\sin x}{=} \int (1-u^2)\,du$$

$\frac{du}{dx} = \cos x \quad \therefore \quad du = (\cos x)\,dx$

$$= u - \frac{1}{3}u^3 + C = \boxed{\sin x - \frac{1}{3}\sin^3 x + C}$$

$\int \cos^2(x)\,dx$ is not as easy! We need integration by parts

$$\int \cos^2(x)\,dx = \int \underbrace{\cos x}_{f}\cdot\underbrace{\cos x}_{g'}\,dx = \underbrace{\cos x}_{f}\cdot\underbrace{\sin x}_{g} - \int \underbrace{(-\sin x)}_{f'}\cdot\underbrace{\sin x}_{g}\,dx$$

Thus:

$$\int \cos^2 x\,dx = \cos x\cdot\sin x + \int \sin^2 x\,dx$$

But $\cos^2 x + \sin^2 x = 1$ so $\sin^2 x = 1 - \cos^2 x$

$$= \cos x \sin x + \int (1-\cos^2 x)\,dx$$

$$\therefore \int \cos^2 x\,dx = \cos x \sin x + \int dx - \int \cos^2 x\,dx$$

move it to the left-hand side

$$\therefore 2\int \cos^2 x\,dx = \cos x\cdot\sin x + x + C$$

$$\therefore \boxed{\int \cos^2 x\,dx = \frac{1}{2}\cos x\cdot\sin x + \frac{1}{2}x + \tilde{C}}$$

Alternative method: use the double angle formula: $\boxed{\cos(2x) = 2\cos^2 x - 1}$

or $\cos^2 x = \dfrac{\cos(2x)+1}{2}$.

Thus:

$$\int \cos^2 x\,dx = \int \left(\frac{1}{2}\cos(2x) + \frac{1}{2}\right)dx =$$

$$= \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + \frac{1}{2}x + C$$

$$= \frac{1}{4}\sin(2x) + \frac{1}{2}x + C$$

$$= \frac{2\sin x\cos x}{4} + \frac{1}{2}x + C = \boxed{\frac{1}{2}\sin x\cos x + \frac{1}{2}x + C}$$