MA 138 – Calculus 2 with Life Science Applications
Rational Functions and Partial Fractions
(Section 7.3)

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Lecture b

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(1)
$$\int \frac{5}{(3x+2)^4} dx \qquad \text{ set } \boxed{u = 3x+2} \qquad \text{so + eat } \frac{du}{dx} = 3$$

$$\boxed{dx = \frac{1}{3} du}$$

hence if we substitute back
$$= \int \frac{5}{u^4} \cdot \frac{1}{3} du = \int \frac{5}{3} u^{-4} du = \frac{5}{3} \cdot \frac{1}{-3} u^{-3} + C$$

$$= -\frac{5}{4} \cdot \frac{1}{u^3} + C = \left[-\frac{5}{4} \cdot \frac{1}{(3x+2)^3} + C \right]$$

(2)
$$\int \frac{2 \times -2}{(x^2 - 2x + 5)^3} dx \quad \text{set} \quad \frac{u = x^2 - 2x + 5}{\frac{du}{dx}} = 2 \times -2 \quad \frac{du}{(2x - 2)dx = du}$$
hence
$$= \int \frac{du}{u^3} = \int u^{-3} du = -\frac{1}{2} u + C = -\frac{1}{2u^2} + C$$

$$= \left[-\frac{1}{2(x^2 - 2x + 5)^2} + C \right]$$

Example 1

Evaluate the following indefinite integrals

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Section 7.3: Rational Functions and Partial Fractions

 \blacksquare A rational function f is the quotient of two polynomials. That is,

$$f(x) = \frac{P(x)}{Q(x)}$$

where P(x) and Q(x) are polynomials.

- To integrate such a function, we write f(x) as a sum of a polynomial and simpler rational functions (=partial-fraction decomposition).
- These simpler rational functions, which can be integrated with the methods we have learned, are of the form

$$\frac{A}{(ax+b)^n}$$
 or $\frac{Bx+C}{(ax^2+bx+c)^n}$

where A, B, C, a, b, and c are constants and n is a positive integer.

■ In this form, the quadratic polynomial $ax^2 + bx + c$ can no longer be factored into a product of two linear functions with real coefficients.

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Proper Rational Functions

■ The rational function f(x) = P(x)/Q(x) is said to be **proper** if the degree of the polynomial in the numerator, P(x), is strictly less than the degree of the polynomial in the denominator, Q(x),

$$f(x) = \frac{P(x)}{Q(x)}$$
 proper \iff deg $P(x) <$ deg $Q(x)$.

■ Which of the following three rational functions

$$f_1(x) = \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x + 5} \qquad f_2(x) = \frac{x}{x + 2} \qquad f_3(x) = \frac{2x - 3}{x^2 + x}$$
 is proper? Only $f_3(x)$ is proper.

■ The first step in the partial-fraction decomposition procedure is to use the long division algorithm to write f(x) as a sum of a polynomial and a **proper** rational function.

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Example 2

Divide the polynomial

$$A(x) = 2x^2 - x - 4$$
 by $B(x) = x - 3$.

(Complete the above table and check your work!)

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Algebra Review

Dividing polynomials is much like the familiar process of dividing numbers. This process is the *long division algorithm for polynomials*.

Long Division Algorithm

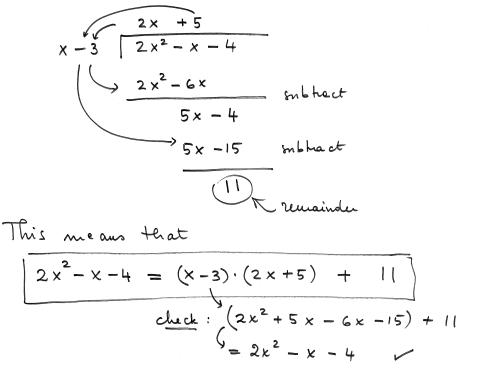
If A(x) and B(x) are polynomials, with $B(x) \neq 0$, then there exist unique polynomials Q(x) and R(x), where R(x) is either 0 or of degree strictly less than the degree of B(x), such that

$$A(x) = Q(x) \cdot B(x) + R(x)$$

The polynomials A(x) and B(x) are called the **dividend** and **divisor**, respectively; Q(x) is the **quotient** and R(x) is the **remainder**.

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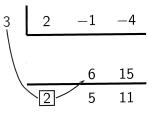


- Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form x-c, where c is a number. In synthetic division we write only the essential part of the long division table.
- In synthetic division we abbreviate the polynomial A(x) by writing only its coefficients.

Moreover, instead of B(x) = x - c, we simply write 'c.' Writing c instead of -c allows us to add instead of subtract!

Example 2 (revisited): Divide

$$A(x) = 2x^2 - x - 4$$
 by $B(x) = x - 3$.

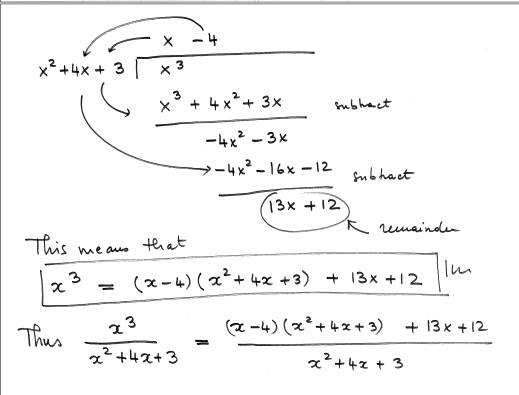


We obtain Q(x) = 2x + 5 and R(x) = 11. That is,

 $2x^2 - x - 4 = (2x + 5)(x - 3) + 11.$

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Example 3 (Online Homework # 3)

Use the Long Division Algorithm to write f(x) as a sum of a polynomial and a proper rational function

$$f(x) = \frac{x^3}{x^2 + 4x + 3}.$$

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Now split the faction on the right hand side as the sum of 2 factions: $\frac{x^{3}}{x^{2}+4x+3} = \frac{(x-4)(x^{2}+4x+3)}{x^{2}+4x+3} + \frac{13x+12}{x^{2}+4x+3}$ rational fraction

Partial Fraction Decomposition (linear factors)

Case of Distinct Linear Factors

Q(x) is a product of m distinct linear factors. Q(x) is thus of the form

$$Q(x) = a(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_m)$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are the *m* distinct roots of Q(x).

The rational function can then be written as

$$\frac{P(x)}{Q(x)} = \frac{1}{a} \left[\frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \dots + \frac{A_m}{x - \alpha_m} \right]$$

We will see in the next examples how the constants A_1, A_2, \ldots, A_m are determined.

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We have shown before that
$$\frac{x^3}{x^2 + 4x + 3} = x - 4 + \frac{13x + 12}{x^2 + 4x + 3}$$

Thus:

$$\int \frac{x^3}{x^2 + 4x + 3} dx = \int (x - 4) dx + \int \frac{13x + 12}{x^2 + 4x + 3} dx$$

$$= \frac{1}{2}x^2 - 4x + C$$

$$let's look at this$$

Notice:
$$\frac{13x+12}{x^2+4x+3} = \frac{13x+12}{(x+1)(x+3)} = \frac{A}{x+3} + \frac{B}{x+1}$$
want
for some constants

Example 3 (cont.d)

Evaluate the indefinite integral: $\int \frac{x^3}{x^2 + 4x + 3} \, dx.$

Note: from the calculations carried out in the first part of the example, we know that our problem reduces to

$$\int (x-4) dx + \int \frac{13x+12}{(x+3)(x+1)} dx.$$

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$$\frac{13x+12}{(x+1)(x+3)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1) + B(x+3)}{(x+3)(x+1)}$$
give common demandments
$$= \frac{Ax + A + Bx + BB}{(x+3)(x+1)} = \frac{(A+B)x + (A+3B)}{(x+1)(x+3)}$$
This means that
$$\frac{13x+(2) = (A+B)x + (A+3B)}{(A+B)x + (A+3B)} = \frac{A}{A+3B=12}$$

$$A+B=13$$

$$A+3B=12$$

$$A+3B=12$$

$$A=13-B$$

$$A=13-B=12-3B$$

This mean that
$$\frac{13 \times +12}{2^2 + 4z + 3} = \frac{27}{2} \cdot \frac{1}{2 + 3} - \frac{1}{2} \cdot \frac{1}{2 + 1}$$

Thus

$$\int \frac{13x+12}{x^2+4x+3} dx = \frac{27}{2} \int \frac{1}{x+3} dx - \frac{1}{2} \int \frac{1}{x+1} dx$$
$$= \frac{27}{2} \ln|x+3| - \frac{1}{2} \ln|x+1| + C$$

Thus:

$$\int \frac{x^3}{x^2+4x+3} = \frac{1}{2}x^2-4x + \frac{27}{2}\ln|x+3|-\frac{1}{2}\ln|x+1|+C$$

(Heaviside) cover-up method

We illustrate this method by using the previous example:

$$\frac{13x+12}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)}$$

$$A(x+1) + B(x+3) = 13x + 12$$
 (*)

Set
$$x = -1$$
 in (*). We obtain
$$A \cdot 0 + B \cdot (-1 + 3) = 13(-1) + 12$$

$$A \cdot (-3 + 1) + 0 = 13(-3) + 12$$

$$A \cdot (-2) = -27$$

$$A \cdot (-2) = -27$$

B = -1/2

A = 27/2

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Example 4 (Online Homework # 8)

Find the integral:
$$\int_2^5 \frac{2}{x^2 - 1} dx.$$

Considu the fraction $\frac{2}{x^2-1} = \frac{2}{(x-1)(x+1)}$

We want to decomposit as:

$$\frac{2}{x^{2}-1} = \frac{A}{x-1} + \frac{B}{x+1} = give common deus un' ustro$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

Thus
$$2 = A(x+1) + B(x-1)$$

evaluate at
$$\boxed{z=1}$$
: $2=2\cdot A+B\cdot 0$.. $\boxed{A=1}$

evaluate at
$$[x=-1]$$
: $2 = A \cdot (0) + B(-2)$ if $B=-1$

Thus:
$$\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$$

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Thus:
$$\int \frac{2}{x^{2}-1} dx = \int \frac{1}{z-1} dz - \int \frac{1}{z+1} dz$$

$$= \ln |z-1| - \ln |z+1| + C$$

$$= \ln \left| \frac{z-1}{z+1} \right| + C$$

Finally:
$$\int_{2}^{5} \frac{2}{x^{2}-1} dx = \lim_{|x-1| \le x+1} \frac{1}{2} = \lim_{$$

Example 5 (Online Homework # 6)

Evaluate the indefinite integral: $\int \frac{1}{x(x+1)} dx.$

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We need to decompose the faction $\frac{1}{x(x+1)}$ as $\frac{\Delta}{x} + \frac{B}{x+1}$. Thus:

$$\frac{1}{\chi(\chi+1)} = \frac{\Delta}{\chi} + \frac{B}{\chi+1} = \frac{\Delta(\chi+1) + B \cdot \chi}{\chi(\chi+1)}$$

1 = A(2+1) + B2 Thus (A+B)x+A=1· evaluate at x=01 = A. 1 + B. 0 : | A=1 | B=-1 · evaluate at \(z = -1 \)

1= A. 0 + B (-1)

No matter which method we choose, we obtain $\frac{1}{x(z+1)} = \frac{1}{x} - \frac{1}{z+1}$ Thus

$$\int \frac{1}{x(x+1)} dx = \int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$

$$= \ln |x| - \ln |x+1| + C$$

$$= \ln \left| \frac{x}{x+1} \right| + C$$