MA 138 – Calculus 2 with Life Science Applications
Improper Integrals
(Section 7.4)

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Improper Integrals

We discuss definite integrals of two types with the following characteristics:

(1) **one or both limits of integration are infinite**; that is, the integration interval is unbounded. For example

$$\int_1^{\infty} e^{-x}\,dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx;$$

(These integrals are very important in Probability and Statistics!)

(2) **the integrand becomes infinite at one or more points of the interval of integration**. For example

$$\int_{-1}^{1} \frac{1}{x^2}\,dx \qquad \text{or} \qquad \int_0^1 \frac{1}{2\sqrt{x}}\,dx.$$

We call such integrals **improper integrals**.

Type 2: Unbounded Integrand

What if the integrand becomes infinite at one or both endpoints of the interval of integration?

- If f is continuous on $(a,b]$ and $\lim\limits_{x\longrightarrow a^+} f(x) = \pm\infty$, we define
$$\int_a^b f(x)\,dx := \lim_{c\longrightarrow a^+} \int_c^b f(x)\,dx$$
provided that this limit exists.
- If f is continuous on $[a,b)$ and $\lim\limits_{x\longrightarrow b^-} f(x) = \pm\infty$, we define
$$\int_a^b f(x)\,dx := \lim_{c\longrightarrow b^-} \int_a^c f(x)\,dx$$
provided that this limit exists.

If the limit does not exist, we say that the integral diverges.

Example 1

Determine whether the improper integral

$$\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$$

is convergent. If the integral is convergent, compute its value.

$$= \int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$$

use the substitution

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int_0^1 \frac{1}{\sqrt{u}}\,du$$

notice that the integrand is not defined when $u=0$!!

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{u}}\,du = \lim_{a\to 0^+} \Big[2\sqrt{u} \Big]_a^1 =$$

$$= \lim_{a\to 0^+} \Big[2\sqrt{1} - 2\sqrt{a} \Big] = 2 - 2\cdot \underbrace{\lim_{a\to 0^+}\sqrt{a}}_{=0}$$

$$= \boxed{2}$$

Example 2 (Problem #28, Section 7.4, page 396)

Determine whether the improper integral

$$\int_1^e \frac{1}{x\ln x}\,dx$$

is convergent. If the integral is convergent, compute its value.

$$= \int_1^e \frac{1}{x\cdot \ln x}\,dx$$

use the substitution $u = \ln x$

$du = \frac{1}{x}\,dx$. Thus

$$= \int_0^1 \frac{1}{u}\,du$$

notice that the integrand $\frac{1}{u}$ is not defined when $u=0$.

$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{u}\,du = \lim_{a\to 0^+} \Big[\ln|u| \Big]_a^1 =$$

$$= \lim_{a\to 0^+} \Big[\underbrace{\ln(1)}_{0} - \ln(a) \Big] =$$

$$= - \lim_{a\to 0^+} \ln(a) = -(-\infty) = +\infty$$

diverges

Example 3 (Online Homework #7)

Determine whether the improper integral

$$\int_0^9 \frac{4}{(x-6)^2}\,dx$$

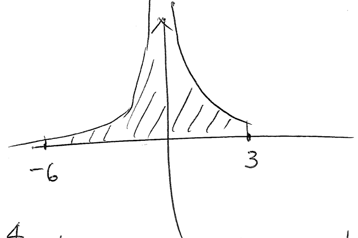
is convergent. If the integral is convergent, compute its value.

$$= \int_0^9 \frac{4}{(x-6)^2}\,dx$$

use the substitution $u = x-6$

so that $du = dx$

$$= \int_{-6}^{3} \frac{4}{u^2}\,du$$



$$= \int_{-6}^{0} \frac{4}{u^2}\,du + \int_{0}^{3} \frac{4}{u^2}\,du$$

the integrand $\frac{1}{u^2}$ is not defined at $u=0$

Note that both integrals are "$+\infty$". So the whole integral does not exist (or diverges).

e.g. $\displaystyle\int_0^3 \frac{4}{u^2}\,du = \lim_{a\to 0^+}\int_a^3 \frac{4}{u^2}\,du = \lim_{a\to 0^+}\left[-\frac{4}{u}\right]_a^3 =$

$$= \lim_{a\to 0^+}\left(-\frac{4}{3} - \left(-\frac{4}{a}\right)\right)$$

$$= \lim_{a\to 0^+}\left[\frac{4}{a} - \frac{4}{3}\right] = +\infty - \frac{4}{3} = \underline{\underline{+\infty}}$$

Similarly:

$$\int_{-6}^{0} \frac{4}{u^2}\,du = \lim_{b\to 0^-}\int_{-6}^{b} \frac{4}{u^2}\,du = \lim_{b\to 0^-}\left[-\frac{4}{u}\right]_{-6}^{b}$$

$$= \lim_{b\to 0^-}\left[-\frac{4}{b} - \left(-\frac{4}{(-6)}\right)\right] = \lim_{b\to 0^-}\left[-\frac{4}{b} - \frac{2}{3}\right]$$

$$= -\left[\lim_{b\to 0^-}\frac{4}{b}\right] - \frac{2}{3} = -(-\infty) - \frac{2}{3} = \underline{\underline{+\infty}}$$

Example 4 (Problem #36, Section 7.4, page 397)

Let p be a positive real number. Show that

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p} & \text{for } 0 < p < 1 \\ \infty & \text{for } p \geq 1. \end{cases}$$

E.g.: $\displaystyle\int_0^1 \frac{1}{x}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ both diverge (as $p = 1, 2$, respectively).

E.g.: $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2$ and $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}$ (as $p = 1/2, 1/3$, respectively).

$$\int_0^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x^p}\,dx = \lim_{a\to 0^+}\int_a^1 x^{-p}\,dx$$

$\boxed{\text{for } p \neq 1}$

$$= \lim_{a\to 0^+}\left[\frac{1}{1-p}\,x^{1-p}\right]_a^1 = \lim_{a\to 0^+}\left[\frac{1}{1-p} - \frac{1}{1-p}\,a^{1-p}\right]$$

$$= \frac{1}{1-p} - \frac{1}{1-p}\underbrace{\lim_{a\to 0^+}\left[a^{1-p}\right]}$$

diverges $\boxed{\infty \text{ for } p>1}$

converges $\boxed{0 \text{ for } 0<p<1}$

$\boxed{\text{for } p = 1}$

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+}\Big[\ln|x|\Big]_a^1 = \lim_{a\to 0^+}\Big[0 - \ln(a)\Big]$$

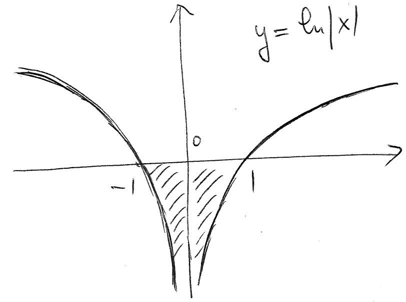
$$= 0 - (-\infty) = \underline{\underline{+\infty}}$$ diverges

Example 5 (Problem #15, Section 7.4, page 396)

Determine whether the improper integral

$$\int_{-1}^{1} \ln|x|\, dx.$$

is convergent. If the integral is convergent, compute its value.



Notice that

$$\int_{-1}^{1} \ln|x|\, dx = \int_{-1}^{0} \ln|x|\, dx + \int_{0}^{1} \ln|x|\, dx$$

If both integrals exist, by symmetry then

$$\int_{-1}^{1} \ln|x|\, dx = 2\int_{0}^{1} \ln x\, dx = \boxed{2 \lim_{a\to 0^+} \int_{a}^{1} \ln x\, dx}$$

Recall that (integration by parts):

$$\int \ln x\, dx = x \cdot \ln x - \int x \cdot \frac{1}{x}\, dx = x \ln x - \int dx$$

$$= x \ln x - x + C = x(\ln x - 1) + C$$

Thus:

$$\int_{-1}^{1} \ln|x|\, dx = 2 \lim_{a\to 0^+} \int_{a}^{1} \ln x\, dx$$

$$= 2 \lim_{a\to 0^+} \Big[x(\ln x - 1) \Big]_{a}^{1} =$$

$$= 2 \lim_{a\to 0^+} \Big\{ \big[1 \cdot (\ln(1) - 1)\big] - \big[a(\ln a - 1)\big] \Big\}$$

$$= 2 \lim_{a\to 0^+} \Big\{ -1 - \underbrace{a \cdot \ln(a)}_{\to 0} + \underbrace{a}_{\to 0} \Big\}$$

$$= \boxed{-2}$$

Note: $\lim_{a\to 0^+} a \cdot \ln(a) = 0 \cdot (-\infty) = \lim_{a\to 0^+} \frac{\ln(a)}{1/a} \overset{\text{l'Hôpital}}{=} \lim_{a\to 0^+} \frac{1/a}{-1/a^2}$

$$= \lim_{a\to 0^+} (-a) = 0$$