MA 138 – Calculus 2 with Life Science Applications
Solving Differential Equations
(Section 8.1)

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Differential Equations ($\equiv$ DEs)

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives.

For example

- $\dfrac{dy}{dx} + 6y = 7$;
- $\dfrac{dy}{dt} + 0.2\,t\,y = 6t$;
- $\dfrac{dP}{dt} = \sqrt{P\,t}$;
- $xy' + y = y^2$.

Differential equations can contain derivatives of any order; for example,

- $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = x\,y$ or $y'' + 6y' - x\,y = 0$

is a DE containing the first and second derivative of the function $y = y(x)$.

If a differential equation contains only the first derivative, it is called a **first-order differential equation**: $\dfrac{dy}{dx} = h(x,y)$.

DEs arise for example in biology (e.g. models of population growth), economics (e.g. models of economic growth), and many other areas.

exponential growth model: $\dfrac{dN}{dt} = rN \qquad N(0) = N_0$;

logistic growth model: $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right) \qquad N(0) = N_0$;

von Bertalanffy models: $\dfrac{dL}{dt} = k(L_\infty - L) \qquad L(0) = L_0$,

$$\frac{dW}{dt} = \eta W^{2/3} - \kappa W \qquad W(0) = W_0;$$

Solow's economic growth model: $\dfrac{dk}{dt} = sk^\alpha - \delta k \qquad k(0) = k_0$.

Example 1

Consider the differential equation $(t+1)\dfrac{dy}{dt} - y + 6 = 0$.

Which of the following functions

$$y_1(t) = t + 7 \qquad y_2(t) = 3t + 21 \qquad y_3(t) = 3t + 9$$

are solutions for all t?

$$\boxed{(t+1)\frac{dy}{dt} - y + 6 = 0}$$

(1) $y_1(t) = t+7$ so $\frac{dy_1}{dt} = 1$

Hence $(t+1)\cdot\frac{dy_1}{dt} - y_1 + 6 \stackrel{?}{=} 0$

becomes $(t+1)(1) - (t+7) + 6 \stackrel{?}{=} 0$

$t+1-t-7+6 = 0$ Yes

(2) $y_2(t) = 3t+21$ so $\frac{dy_2}{dt} = 3$

Hence $(t+1)\cdot\frac{dy_2}{dt} - y_2 + 6 \stackrel{?}{=} 0$

becomes $(t+1)(3) - (3t+21) + 6 \stackrel{?}{=} 0$

$\cancel{3t} + 3 - \cancel{3t} - 21 + 6 \stackrel{?}{=} 0$

$-12 = 0$ NO

(3) $y_3(t) = 3t+9$ so $\frac{dy_3}{dt} = 3$

Hence $(t+1)\frac{dy_3}{dt} - y_3 + 6 \stackrel{?}{=} 0$

becomes

$(t+1)(3) - (3t+9) + 6 \stackrel{?}{=} 0$

$3t+3-3t-9+6 \stackrel{?}{=} 0$

$0 = 0$ YES

Hence $y_1(t)$ and $y_3(t)$ are solutions but $y_2(t)$ is not a solution

Separable Differential Equations

We will restrict ourselves to first-order differential equations

$$\frac{dy}{dx} = h(x,y) \qquad \text{of the form} \qquad \frac{dy}{dx} = f(x)g(y).$$

That is, the right-hand side of the equation is the product of two functions, one depending only on x, $f(x)$, the other only on y, $g(y)$.

Such equations are called **separable differential equations**.

This type of differential equations includes two special cases:

- **pure-time differential equations**: $\frac{dy}{dx} = f(x)$ [i.e., $g(y) \equiv 1$]
- **autonomous differential equations**: $\frac{dy}{dx} = g(y)$ [i.e., $f(x) \equiv 1$]

  (DEs of this form are frequently used in biological models.)

In order to solve the separable differential equation

$$\boxed{\frac{dy}{dx} = f(x)g(y)}, \qquad (*)$$

we divide both sides of $(*)$ by $g(y)$ [assuming that $g(y) \neq 0$]:

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x).$$

Now, if $y = u(x)$ is a solution of $(*)$, then $u(x)$ satisfies

$$\frac{1}{g[u(x)]}u'(x) = f(x).$$

If we integrate both sides with respect to x, we find that

$$\int \frac{1}{g[u(x)]}u'(x)\,dx = \int f(x)\,dx \qquad \text{or} \qquad \boxed{\int \frac{1}{g(y)}\,dy = \int f(x)\,dx}$$

since $g[u(x)] = g(y)$ and $u'(x)dx = dy$.

Example 1 (again)

Solve the differential equation $(t+1)\dfrac{dy}{dt} - y + 6 = 0$.

$$(t+1)\frac{dy}{dt} - y + 6 = 0 \longleftrightarrow (t+1)\frac{dy}{dt} = y - 6$$

Separate variables and integrate

$$\frac{dy}{y-6} = \frac{dt}{t+1} \longleftrightarrow \int \frac{1}{y-6}\,dy = \int \frac{1}{t+1}\,dt$$

$$\Longrightarrow \ln(y-6) = \ln(t+1) + C$$

take the exponential of both sides

$$e^{\ln(y-6)} = e^{\ln(t+1)+C} = e^{\ln(t+1)} \cdot e^{C}$$

$$\therefore \quad y - 6 = (t+1)A \qquad \text{where } \boxed{A = e^{C}}$$

$$\therefore \quad \boxed{y = A(t+1) + 6}$$

If you prefer we can write

$$y = A \cdot t + A + 6$$

Thus for $A = 1 \leadsto y_1(t) = t + 7$

for $A = 3 \leadsto y_3(t) = 3t + 9$

But we cannot get $y_2(t)$.

Example 2 (Online Homework # 2)

Solve the following initial value problem

$$\frac{dy}{dt} + 0.2ty = 6t$$

with $y(0) = 4$.

$$\frac{dy}{dt} + 0.2\,ty = 6t \iff \frac{dy}{dt} = 6t - 0.2ty$$

$$\frac{dy}{dt} = 0.2t(30-y) = -0.2t(y-30)$$

Separate and integrate:

$$\frac{dy}{y-30} = -0.2t\,dt \iff \int \frac{1}{y-30}\,dt = \int -0.2t\,dt$$

$$\Longrightarrow \ln(y-30) = -0.1t^2 + C \qquad \text{take } \underline{\underline{\text{exp.}}}$$

$$e^{\ln(y-30)} = e^{-0.1t^2+C}$$

$$\Longrightarrow \boxed{y - 30 = A \cdot e^{-0.1t^2}}$$

$$\therefore\ y = 30 + Ae^{-0.1t^2}$$

Use the initial condition $y(0) = 4$

$$\therefore\ 4 = 30 + A\underbrace{e^{-0.1(0)^2}}_{1} \qquad \therefore\ \boxed{A = -26}$$

$$\therefore\ \boxed{y = 30 - 26e^{-0.1t^2}}$$

Example 3 (Online Homework # 3)

Find the solution of the differential equation

$$\frac{dP}{dt} = \sqrt{P\,t}$$

that satisfies the initial condition $P(1) = 7$.

$$\frac{dP}{dt} = \sqrt{Pt} \qquad \text{with } P(1) = 7$$

$$\frac{dP}{dt} = \sqrt{P}\cdot\sqrt{t} \Longrightarrow \frac{1}{\sqrt{P}}\,dP = \sqrt{t}\,dt$$

(separate variables)

$$\int P^{-1/2}\,dP = \int t^{1/2}\,dt \Longrightarrow \boxed{2P^{1/2} = \frac{2}{3}t^{3/2} + C}$$

(integrate)

(simplify by 2 on both sides)

$$\boxed{P^{1/2} = \frac{1}{3}t^{3/2} + \tilde{C}} \qquad \tilde{C} = \frac{C}{2} \text{ constant}$$

Now use the initial condition.

$$P^{1/2} = \tfrac{1}{3} t^{3/2} + \tilde{C} \qquad P(1) = 7$$

$$\therefore \quad \sqrt{7} = \tfrac{1}{3}(1)^{3/2} + \tilde{C} \qquad \therefore \quad \tilde{C} = \sqrt{7} - \tfrac{1}{3}$$

Hence :

$$\boxed{P(t) = \left(\tfrac{1}{3} t^{3/2} + \sqrt{7} - \tfrac{1}{3}\right)^2}$$

$$= \tfrac{1}{9}\left(t\sqrt{t} + 3\sqrt{7} - 1\right)^2$$

Example 4 (Online Homework # 5)

Find the solution of the differential equation

$$xy' + y = y^2$$

that satisfies the initial condition $y(1) = -1$.

$$x y' + y = y^2 \qquad y(1) = -1$$

$$x\left(\frac{dy}{dx}\right) = y^2 - y \quad \longleftrightarrow \quad \boxed{\frac{1}{y^2 - y}\, dy = \frac{1}{x}\, dx}$$

(separate variables)

now integrate

$$\int \underbrace{\frac{1}{y(y-1)}}\, dy = \int \frac{1}{x}\, dx$$

partial fractions.... check that

$$\boxed{\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y}}$$

Thus :

$$\int \left(\frac{1}{y-1} - \frac{1}{y}\right) dy = \int \frac{1}{x}\, dx$$

$$\therefore \quad \ln(y-1) - \ln y = \ln x + C$$

$$\therefore \quad \ln\left(\frac{y-1}{y}\right) = \ln x + C$$

Take exponential of both sides

$$\frac{y-1}{y} = A \cdot x \qquad \text{where } A = e^C$$

initial condition

$$\therefore \quad y - 1 = A x y \; ; \quad \text{when } \boxed{x = 1 \quad y = -1}$$

$$\text{So} \quad -2 = A(1)(-1) \qquad \therefore \quad A = 2$$

Thus $y - 1 = 2xy$

$\Longrightarrow \quad y - 2xy = 1$

$y(1-2x) = 1$

$$\boxed{y = \frac{1}{1-2x}}$$

solution to our IVP

Pure-Time Differential Equations

In many applications, the independent variable represents time. If the rate of change of a function depends only on time, we call the resulting differential equation a **pure-time differential equation**. Such a differential equation is of the form

$$\frac{dy}{dx} = f(x), \quad x \in I, \qquad y(x_0) = y_0,$$

where I is an interval and x represents time; the number x_0 is in the interval I.

The solution can then be written as

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(u)\, du.}$$

Example 5 (Example # 1, Section 8.1, p. 430)

Suppose that the volume $V(t)$ of a cell at time t changes according to

$$\frac{dV}{dt} = \sin t \qquad \text{with} \qquad V(0) = 3.$$

Find $V(t)$.

$\frac{dV}{dt} = \sin t \quad$ with $\quad V(0) = 3$

$$V(t) = V(0) + \int_0^t \frac{dV}{du}\, du$$

$$= 3 + \int_0^t \sin(u)\, du$$

$$= 3 + \Big[-\cos(u)\Big]_0^t =$$

$$= 3 + \big[-\cos(t) + \cos(0)\big]$$

$$= \boxed{4 - \cos(t)}$$

Autonomous Differential Equations

Many of the differential equations that model biological situations are of the form

$$\frac{dy}{dx} = g(y)$$

where the right-hand side does not explicitly depend on x. These equations are called **autonomous differential equations**.

Formally, we can solve this autonomous differential equation by separation of variables. We begin by dividing both sides of the equation by $g(y)$ and multiplying both sides by dx, to obtain

$$\frac{1}{g(y)} dy = dx.$$

Integrating both sides then gives $\int \frac{1}{g(y)} dy = \int dx.$

Example 6 (Online Homework # 1)

Find the particular solution of the differential equation

$$\frac{dy}{dx} + 6y = 7$$

satisfying the initial condition $y(0) = 0$.

$\frac{dy}{dx} + 6y = 7$ $\qquad y(0) = 0$

$\frac{dy}{dx} = 7 - 6y \quad \longleftrightarrow \quad \frac{1}{7-6y} dy = dx$

$\int \frac{6}{6y-7} dy = \int (-6) dx$ $\qquad$ Thus

$\ln(6y-7) = -6x + C$ $\qquad$ Take exp

$6y - 7 = A \cdot e^{-6x}$ $\qquad$ when $A = e^C$

$\boxed{6y = 7 + Ae^{-6x}}$ (u

when $x=0$ then $y=0$ is our initial condition.

So from $6y = 7 + Ae^{-6x}$ we get

$0 = 7 + A\underbrace{e^0}_{1}$ $\quad \therefore \boxed{A = -7}$

$\therefore 6y = 7 - 7e^{-6x}$ or

$\boxed{y = \frac{7}{6}(1 - e^{-6x})}$

Example 7 (Problem # 39, Section 8.1, p. 440)

Find the general solution of the differential equation $\frac{dy}{dx} = y^2 - 4$.

$$\frac{dy}{dx} = y^2 - 4 \Longrightarrow \frac{dy}{y^2-4} = dx$$

Hence after we integrate:

$$\int \frac{1}{y^2-4}\,dy = \int dx$$

Note that $\frac{1}{y^2-4} = \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2}$

$$= \frac{A(y+2) + B(y-2)}{(y-2)(y+2)}$$

So that we want A and B such that

$$1 = A(y+2) + B(y-2)$$

if $\boxed{y=2}$ then $1 = 4\cdot A$; if $\boxed{y=-2}$ then $1 = B(-4)$

Thus $\frac{1}{y^2-4} = \frac{1}{4}\left[\frac{1}{y-2} - \frac{1}{y+2}\right]$

$$\therefore \int \frac{1}{4}\left[\frac{1}{y-2} - \frac{1}{y+2}\right] dy = \int dx \quad \text{or}$$

$$\int \left(\frac{1}{y-2} - \frac{1}{y+2}\right) dy = \int 4\,dx$$

$$\ln(y-2) - \ln(y+2) = 4x + C$$

or $\ln\left(\frac{y-2}{y+2}\right) = 4x + C$ Take exp.

$$\frac{y-2}{y+2} = e^{4x}\cdot A \quad \text{where } A = e^C$$

$$y - 2 = Ae^{4x}\cdot(y+2)$$

$$y - yAe^{4x} = 2Ae^{4x} + 2$$

$$y(1 - Ae^{4x}) = 2(1 + Ae^{4x})$$

$$\therefore \boxed{y = 2\,\frac{1 + Ae^{4x}}{1 - Ae^{4x}}}$$

Multiply top and bottom by e^{-4x}

$$\boxed{y = 2\,\frac{e^{-4x} + A}{e^{-4x} - A}}$$

lim

$$\lim_{x\to\infty} y(x) = 2\,\frac{0+A}{0-A} = -2$$

So