MA 138 – Calculus 2 with Life Science Applications
Linear Maps
(Section 9.3)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Outline

- We mostly focus on 2×2 matrices, but point out that we can generalize our discussion to arbitrary $n \times n$ matrices.
- Consider a map of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or, in short,} \qquad \mathbf{v} \mapsto A\mathbf{v}$$

  where A is a 2×2 matrix and $\mathbf{v}$ is a 2×1 (column) vector.
- Since $A\mathbf{v}$ is a 2×1 vector, this map takes a 2×1 vector and maps it into a 2×1 vector. This enables us to apply A repeatedly: We can compute $A(A\mathbf{v}) = A^2\mathbf{v}$, which is again a 2×1 vector, and so on.
- We will **first** look at vectors $\mathbf{v}$, **then** at maps $\mathbf{v} \mapsto A\mathbf{v}$, and **finally** at iterates of the map A (i.e., $A^2\mathbf{v}$, $A^3\mathbf{v}$, and so on).

Graphical Representation of (Column) Vectors

We assume that $\mathbf{v} = \begin{bmatrix} x_{\mathbf{v}} \\ y_{\mathbf{v}} \end{bmatrix}$ is a 2×1 matrix.

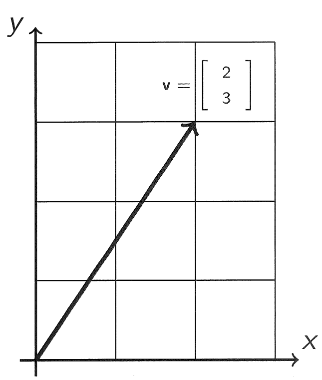
We call $\mathbf{v}$ a column vector or simply a **vector**.

Since a 2×1 matrix has just two components, we can represent a vector in the plane.

For instance, to represent the vector

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

in the x-y plane, we draw an arrow from the origin $(0, 0)$ to the point $(2, 3)$.



Addition of Vectors

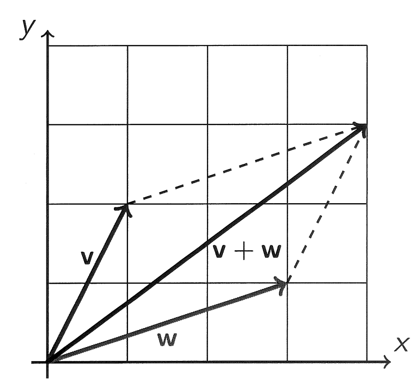
Because vectors are matrices, we can add vectors using matrix addition.

For instance,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

This vector sum has a simple geometric representation. The sum $\mathbf{v} + \mathbf{w}$ is the diagonal in the parallelogram that is formed by the two vectors $\mathbf{v}$ and $\mathbf{w}$.

The rule for vector addition is therefore referred to as the **parallelogram law**.



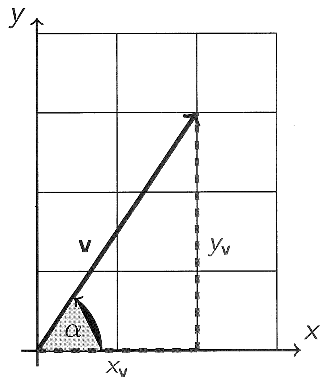
Length of Vectors

The length of the vector $\mathbf{v} = \begin{bmatrix} x_{\mathbf{v}} \\ y_{\mathbf{v}} \end{bmatrix}$, denoted by $|\mathbf{v}|$, is the distance from the origin $(0,0)$ to the point $(x_{\mathbf{v}}, y_{\mathbf{v}})$.

By Pythagoras Theorem we have

$$\text{length of } \mathbf{v} = \|\mathbf{v}\| = \sqrt{x_{\mathbf{v}}^2 + y_{\mathbf{v}}^2}$$

We define the direction of **v** as the angle α between the positive x-axis and the vector **v**. The angle α is in the interval $[0, 2\pi)$ and satisfies $\tan \alpha = y_{\mathbf{v}}/x_{\mathbf{v}}$.



We thus have two distinct ways of representing vectors in the plane: We can use

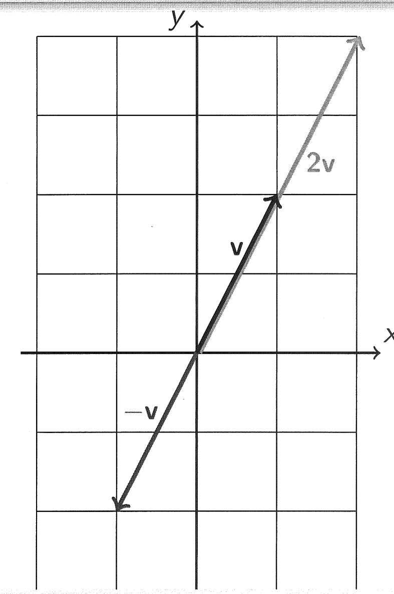
- either the endpoint $(x_{\mathbf{v}}, y_{\mathbf{v}})$
- or the length and direction $(\|\mathbf{v}\|, \alpha)$.

Scalar Multiplication of Vectors

Multiplication of a vector by a scalar is carried out componentwise.

If we multiply $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by 2, we get $2\mathbf{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. This operation corresponds to changing the length of the vector by the factor 2.

If we multiply $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ by -1, then the resulting vector is $-\mathbf{v} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, which has the same length as the original vector, but points in the opposite direction.



Linear Maps (also called Linear Transformations)

We start with a graphical approach to study maps of the form

$$\mathbf{v} \mapsto A\mathbf{v}$$

where A is a 2×2 matrix and **v** is a 2×1 vector.

Since $A\mathbf{v}$ is a 2×1 vector as well, the map A takes the 2×1 vector **v** and maps it to the 2×1 vector $A\mathbf{v}$ can be thought of as a map from the plane $\mathbb{R}^2$ to the plane $\mathbb{R}^2$.

We will discuss simple examples of maps from $\mathbb{R}^2$ into $\mathbb{R}^2$ defined by $\mathbf{v} \mapsto A\mathbf{v}$, that take the vector **v** and rotate, stretch, or contract it.

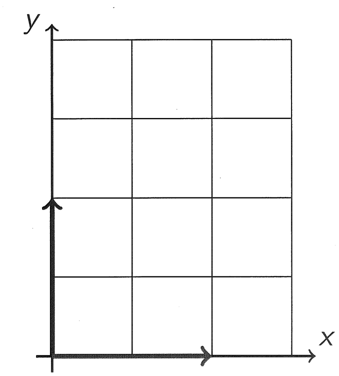
For an arbitrary matrix A, vectors may be moved in a way that has no simple geometric interpretation.

Example 1 (Reflections)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

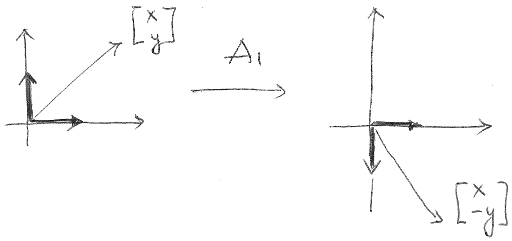


$A_1 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$

$A_1\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$A_1\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$A_1\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\-y\end{bmatrix}$



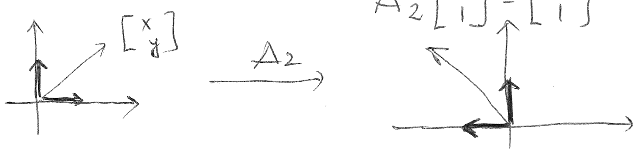
∴ A_1 is a reflexion about the x-axis

$A_2 = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$

$A_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$

$A_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$A_2\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\y\end{bmatrix}$



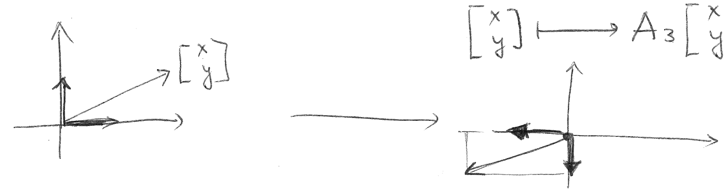
∴ A_2 is a reflexion about the y-axis

$A_3 = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix} \longmapsto A_3\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix} \longmapsto A_3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto A_3\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-x\\-y\end{bmatrix}$



A_3 is $-I_2$ that is it sends every vector into its opposite.

Finally, A_4: $\begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}0\\-1\end{bmatrix}$; $\begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}-1\\0\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \begin{bmatrix}-y\\-x\end{bmatrix} = -\begin{bmatrix}y\\x\end{bmatrix}$

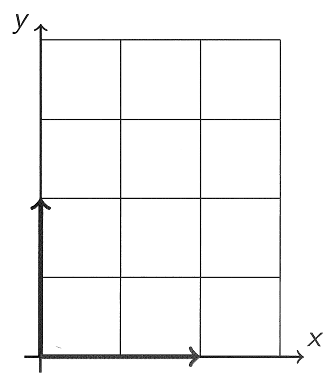
i.e. we reflect a vector about the $y=x$ line and then we send it to its opposite.

Example 2 (Contractions or Expansions)

Describe how multiplication by the matrices below changes the vectors in the picture:

$A_1 = \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$ $A_2 = \begin{bmatrix}1 & 0\\ 0 & 1/2\end{bmatrix}$

$A_3 = \begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix}$ $A_4 = \begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$



A_1: $\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

$\begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix}$

$\begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\y\end{bmatrix}$

that is A_1 doubles the x-coordinate of a vector (it is an expansion in the x-direction)

A_2: it is a contraction by a factor of 2 in the y-direction. I.e.

$\begin{bmatrix}1\\0\end{bmatrix} \longmapsto \begin{bmatrix}1\\0\end{bmatrix}$; $\begin{bmatrix}0\\1\end{bmatrix} \longmapsto \begin{bmatrix}0\\1/2\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix} \longmapsto \begin{bmatrix}x\\y/2\end{bmatrix}$

A_3 multiplies the x-component of a vector by a factor "a". It all depends if a is positive or negative

It also depends whether $|a|<1$ (contraction) or $|a|>1$ (expansion).

A_4 : the x component of a vector is sent into ax

the y-component of a vector is sent into bx

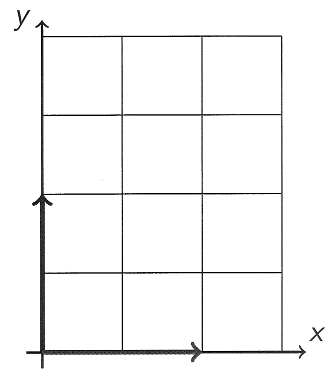
$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} ax \\ by \end{bmatrix}$$

Example 3 (Shears)

Describe how multiplication by the matrices below changes the vectors in the picture:

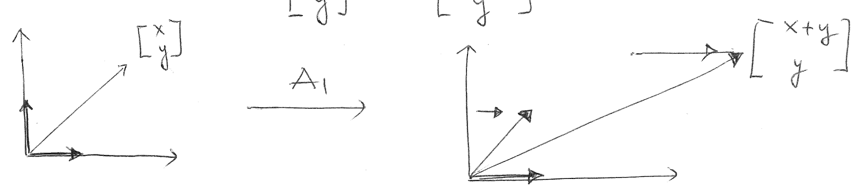
$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$$



$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x+y \\ y \end{bmatrix}$$



they y-component of the vector remains the same whereas the x-component of the vector gets moved to the right of y units.

Same as A_1, but

$$A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

shift in the x component

suppose $a>0$

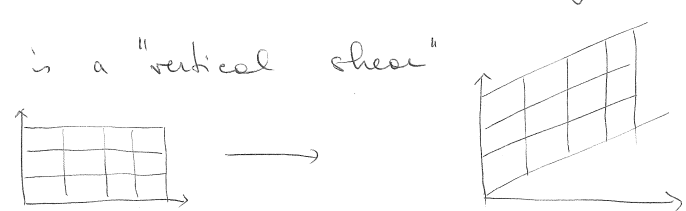
$$A_3 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - ay \\ y \end{bmatrix}$$

negative shift so the components in moved to the left if $a>0$

$$A_4 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ bx + y \end{bmatrix}$$

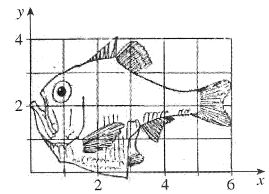
it is now the y-component of the vector that gets changed by adding.

There is a "vertical shear"

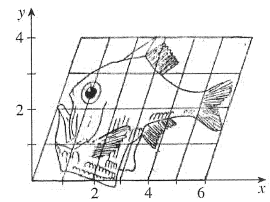


Example 4

Sir D'Arcy Wentworth Thompson (May 2, 1860 - June 21, 1948) was a Scottish biologist, mathematician, and classics scholar. He was a pioneer of mathematical biology. Thompson is remembered as the author of the distinctive 1917 book *On Growth and Form*. The book led the way for the scientific explanation of morphogenesis, the process by which patterns are formed in plants and animals.



Argyropelecus offers



Sternoptyx diaphana

For example, Thompson illustrated the transformation of *Argyropelecus offers* into *Sternoptyx diaphana* by applying a 20° shear mapping ($\equiv$ transvection). What is the form of the matrix that describes this change?

(source: WIKIPEDIA)

From the discussion in Example 3 there is a horizontal shear.

the y-components of the vectors are unchanged but we shift by a factor ay the x component of the vectors.

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : \begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x+ay \\ y \end{bmatrix}$$

$$a = \tan(20^\circ) = 0.36397$$

$$\therefore \begin{bmatrix} 1 & 0.36397 \\ 0 & 1 \end{bmatrix}$$

Rotations

The following matrix rotates a vector in the x-y plane by an angle α:

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

If $\alpha > 0$ the rotation is counterclockwise; if $\alpha < 0$ it is clockwise.

Properties of Rotations:

- $\det(R_\alpha) = \cos^2\alpha + \sin^2\alpha = 1$.
- A rotation by an angle α followed by a rotation by an angle β should be equivalent to a single rotation by a total angle $\alpha + \beta$. In fact, using the usual trigonometric identities, we have

$$\begin{aligned} R_\alpha R_\beta &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R_{\alpha+\beta} \end{aligned}$$

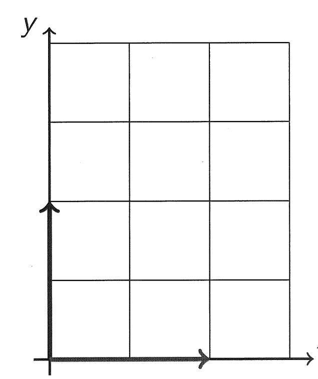
- The previous identity shows that the product of rotations is commutative: $R_\alpha R_\beta = R_\beta R_\alpha$.

Example 5 (Rotations)

Describe how multiplication by the matrices below changes the vectors in the picture:

$$A = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

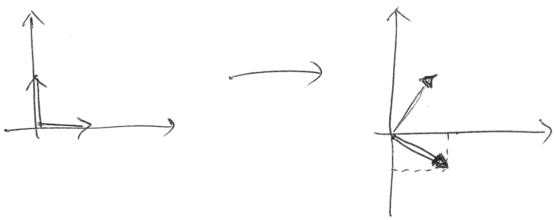
$$B = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$



$$A = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} \sqrt{3}/2 \\ -1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$



clock wise rotation of $\frac{\pi}{6}$ $(= 30°)$

or counter clock wise rotation of $-\frac{\pi}{6}$ $(= -30°)$

$$B = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \longmapsto \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

counter clock wise rotation of $\frac{\pi}{3}$ $(= 60°)$

Properties of Linear Maps

According to the properties of matrix multiplication, the map $\mathbf{v} \mapsto A\mathbf{v}$ satisfies the following conditions:

- $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$, and
- $A(\lambda \mathbf{v}) = \lambda(A\mathbf{v})$, where λ is a scalar.

Because of these two properties, we say that the map $\mathbf{v} \mapsto A\mathbf{v}$ is **linear**.

Example 6 (Problem # 2, Section 9.3, p 533)

Show by direct calculation that $A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$ and $A(\lambda\mathbf{v}) = \lambda(A\mathbf{v})$.

$A(\mathbf{v} + \mathbf{w}) = A\mathbf{v} + A\mathbf{w}$

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x' \\ y' \end{bmatrix}\right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x + x' \\ y + y' \end{bmatrix} \\
&= \begin{bmatrix} a(x+x') + b(y+y') \\ c(x+x') + d(y+y') \end{bmatrix} \\
&= \begin{bmatrix} (ax+by) + (ax'+by') \\ (cx+dy) + (cx'+dy') \end{bmatrix} \\
&= \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} + \begin{bmatrix} ax'+by' \\ cx'+dy' \end{bmatrix} \\
&= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}
\end{aligned}$$

$A(\lambda\mathbf{v}) = \lambda(A\mathbf{v})$

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\lambda\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} = \begin{bmatrix} a(\lambda x) + b(\lambda y) \\ c(\lambda x) + d(\lambda y) \end{bmatrix} \\
&= \begin{bmatrix} \lambda(ax+by) \\ \lambda(cx+dy) \end{bmatrix} = \lambda\begin{bmatrix} ax+by \\ cx+dy \end{bmatrix} \\
&= \lambda\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)
\end{aligned}$$

Example 7

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find $A\mathbf{u}$ and $A\mathbf{v}$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

- $\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\underline{u}} \longmapsto \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- $\underbrace{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}_{\underline{v}} \longmapsto \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix} = 4\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

That is $\underline{u}$ and $\underline{v}$ were special vectors such that $A\underline{u} = -\underline{u}$ $\quad A\underline{v} = 4\underline{v}$

Want to find $\boxed{A\underline{w} = \lambda \underline{w}}$ when possible!

Composition of Linear Maps $\equiv$ Product of Matrices

Consider two linear maps $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$ given by the matrices A_f and A_g

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_f}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} \mapsto \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{A_g}\begin{bmatrix} x' \\ y' \end{bmatrix}$$

That is the coordinates are transformed according to the rules

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \qquad \begin{cases} x'' = \alpha x' + \beta y' \\ y'' = \gamma x' + \delta y' \end{cases}$$

If we compose the two maps we obtain the transformation

$$\begin{cases} x'' = \alpha(ax + by) + \beta(cx + dy) = (\alpha a + \beta c)x + (\alpha b + \beta d)y \\ y'' = \gamma(ax + by) + \delta(cx + dy) = (\gamma a + \delta c)x + (\gamma b + \delta d)y \end{cases}$$

whose matrix representation corresponds to the product $A_g A_f$ of the two matrices

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x'' \\ y'' \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{bmatrix}}_{A_{g \circ f}}\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}}_{A_g}\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A_f}\begin{bmatrix} x \\ y \end{bmatrix}$$