MA 138 – Calculus 2 with Life Science Applications
Linear Systems: Theory
(Section 11.1)

Alberto Corso
⟨alberto.corso@uky.edu⟩

Department of Mathematics
University of Kentucky

Systems of Differential Equations

- Suppose that we are given a set of variables $x_1, x_2, \ldots, x_n$, each depending on an independent variable, say, t, so that

$$x_1 = x_1(t), \; x_2 = x_2(t), \; \ldots, \; x_n = x_n(t).$$

- Suppose also that the dynamics of the variables are linked by n differential equations ($\equiv$DEs) of the first-order; that is,

$$\begin{cases} \dfrac{dx_1}{dt} = g_1(t, x_1, x_2, \ldots, x_n) \\ \quad\vdots \\ \dfrac{dx_n}{dt} = g_n(t, x_1, x_2, \ldots, x_n) \end{cases}$$

- This set of equations is called **a system of differential equations**.
- On the LHS are the derivatives of $x_i(t)$ with respect to t. On the RHS is a function g_i that depends on the variables $x_1, x_2, \ldots, , x_n$ and on t.

Examples

- **Kermack & McKendrick epidemic disease model (SIR, 1927)**

$$\begin{cases} \dfrac{dS}{dt} = -b\,SI \\ \dfrac{dI}{dt} = b\,SI - a\,I \\ \dfrac{dR}{dt} = a\,I \end{cases}$$

$S = S(t) =$ # of susceptible individuals
$I = I(t) =$ # of infected individuals
$R = R(t) =$ # of removed individuals ($\equiv$no longer susceptible)
$a, b =$ constant rates

- **Lotka-Volterra predator-prey model (1910/1920):**

$$\begin{cases} \dfrac{dN}{dt} = r\,N - a\,PN \\ \dfrac{dP}{dt} = ab\,PN - d\,P \end{cases}$$

$N = N(t) =$ prey density
$P = P(t) =$ predator density
$r =$ intrinsic rate of increase of the prey
$a =$ attack rate
$b =$ efficiency rate of predators in turning preys into new offsprings
$d =$ rate of decline of the predators

Direction Field of a System of 2 Autonomous DEs

- Review the notion of the direction field of a DE of the first order $dy/dx = f(x, y)$. We encountered this notion just before Section 8.2 (Handout; Lectures 15 & 16).
- Consider, now a system of two autonomous differential equations

$$\begin{cases} \dfrac{dx}{dt} = g_1(x, y) \\ \dfrac{dy}{dt} = g_2(x, y) \end{cases}$$

- Assuming that y is also a function of x and using the chain rule, we can eliminate t and obtain the DE

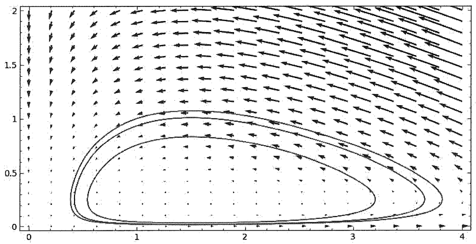
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g_2(x, y)}{g_1(x, y)}$$

of which we can plot the direction field.

Example (Lotka-Volterra)

Consider the system of DEs $\frac{dx}{dt} = x - 4xy$ and $\frac{dy}{dt} = 2xy - 3y$.

The direction field of the differential equation $\frac{dy}{dx} = \frac{(2x-3)y}{x(1-4y)}$ has been produced with the SAGE commands in Chapter 8.



Notice that the trajectories are closed curves. Furthermore, they all seem to revolve around the point $P(3/2, 1/4)$. This is the point where the factors $2x - 3$ and $1 - 4y$ of dy/dt and dx/dt, respectively, are both zero.

Linear Systems of Differential Equations (11.1)

- We first look at the case when the g_i's are linear functions in the variables $x_1, x_2, \ldots, x_n$ — that is,

$$\begin{cases} \frac{dx_1}{dt} &= a_{11}(t)x_1 + \ldots + a_{1n}(t)x_n + f_1(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + \ldots + a_{nn}(t)x_n + f_n(t) \end{cases}$$

- We can write the linear system in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & \ldots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \ldots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

and we call it an **inhomogeneous system of linear, first-order differential equations**.

- We can write our inhomogeneous system of linear, first-order differential equations as follows

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{f}(t)$$

- We are mainly interested in the case when $\mathbf{f}(t) = \mathbf{0}$, that is,

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x},$$

an **homogeneous** system of linear, first-order differential equations.

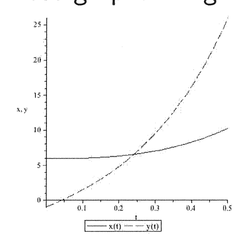
- Finally, we will study the case in which $A(t)$ does not depend on t

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x},$$

an **homogeneous system of linear, first-order differential equations with constant coefficients**.

Example 1 (Problem #8, Exam 3, Spring 2013)

(a) Verify that the functions $x(t) = e^{4t} + 5e^{-t}$ and $y(t) = 4e^{4t} - 5e^{-t}$ (whose graphs are given below) are solutions of the system of DEs



$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = 4x + 3y \end{cases}$$

with $x(0) = 6$ and $y(0) = -1$.

(b) Rewrite the given system of DEs and its solutions in the form

$$\underbrace{\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{system of differential equations}} \qquad \underbrace{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} e^{4t} + 5\begin{bmatrix} \gamma \\ \delta \end{bmatrix} e^{-t}}_{\text{solutions}}$$

for appropriate choices of the constants $a, b, c, d, \alpha, \beta, \gamma$, and δ.

$$\begin{cases} \dfrac{dx}{dt} = y, & x(0) = 6 \\ \dfrac{dy}{dt} = 4x+3y, & y(0) = -1 \end{cases}$$

Proposed solutions

$$x(t) = e^{4t} + 5e^{-t}$$
$$y(t) = 4e^{4t} - 5e^{-t}$$

Let's check that the proposed solutions are indeed satisfying the given system of linear DEs.

$$x(t) = e^{4t} + 5e^{-t} \longrightarrow \frac{dx}{dt} = 4e^{4t} - 5e^{-t}$$

$$y(t) = 4e^{4t} - 5e^{-t} \rightsquigarrow \frac{dy}{dt} = 4(4e^{4t}) - 5(-e^{-t}) = 16e^{4t} + 5e^{-t}$$

Thus

(1) $4e^{4t} - 5e^{-t} = \dfrac{dx}{dt} \stackrel{?}{=} y = 4e^{4t} - 5e^{-t}$ YES ✓

(2) $16e^{4t} + 5e^{-t} = \dfrac{dy}{dt} \stackrel{?}{=} 4x+3y = 4(e^{4t} + 5e^{-t}) + 3(4e^{4t} - 5e^{-t})$
$= 4e^{4} + 20e^{-t} + 12e^{4t} - 15e^{-t}$
$= 16e^{4t} + 5e^{-t}$ YES ✓

We also need to check that the initial conditions are satisfied

$$\begin{cases} x(t) = e^{4t} + 5e^{-t} & \text{when } t=0 & x(0) = e^{0} + 5e^{0} = 6 \\ y(t) = 4e^{4t} - 5e^{-t} & \text{when } t=0 & y(0) = 4e^{0} - 5e^{0} = -1 \end{cases}$$

YES ✓

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 4x+3y \end{cases} \iff \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x(t) = e^{4t} + 5e^{-t} \\ y(t) = 4e^{4t} - 5e^{-t} \end{cases} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^{4t} + 5e^{-t} \\ 4e^{4t} - 5e^{-t} \end{bmatrix} = \begin{bmatrix} e^{4t} \\ 4e^{4t} \end{bmatrix} + \begin{bmatrix} 5e^{-t} \\ -5e^{-t} \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{4t} + 5\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

is the unique solution to

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

NOTICE:

$\begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{4t}$ is made of....? $\qquad \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$ is made of ...? $\qquad \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

eigenvectors and eigenvalues of the matrix

Specific Solutions of a Linear System of DEs

- Consider the system $\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$.
- We claim that the vector-valued function

$$\mathbf{x}(t) = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

where λ, v_1 and v_2 are constants, is a solution of the given system of DEs, for an appropriate choice of values for λ, v_1, and v_2.

- More precisely, $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector of the matrix A corresponding to the eigenvalue λ of A.

$$\underline{x}(t) = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$$

Its derivative is

$$\frac{d\underline{x}}{dt} = \begin{bmatrix} v_1 \lambda e^{\lambda t} \\ v_2 \lambda e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

Thus if we want it to be a solution of

$\frac{d\underline{x}}{dt} = A\underline{x}$ we need

$$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t} = A \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

Cancel $e^{\lambda t}$ on both sides ... we get

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

i.e. λ = eigenvalue, $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ = eigenvector

The Superposition Principle

Principle

Suppose that

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

If $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ and $\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$ are solutions of the given system of DEs, THEN

$$\mathbf{x}(t) = c_1\mathbf{y}(t) + c_2\mathbf{z}(t)$$

is also a solution of the given system of DEs for any constants c_1 and c_2.

$$\frac{d}{dt}(\underline{x}(t)) = \frac{d}{dt}(c_1 \underline{y}(t) + c_2 \underline{z}(t))$$

$$= c_1 \frac{d}{dt}(\underline{y}(t)) + c_2 \frac{d}{dt}(\underline{z}(t))$$

$$= c_1 A\underline{y} + c_2 A\underline{z}$$ ← as both $\underline{y}$ and $\underline{z}$ are solutions

$$= A(c_1\underline{y} + c_2\underline{z})$$ ← properties of matrix multiplication

$$= A(\underline{x}(t))$$

Thus $x(t)$ also satisfies the system of differential equations

The General Solution

Theorem

Let

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

where A is a 2×2 matrix with **two real and distinct eigenvalues** λ_1 and λ_2 with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

THEN

$$x(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

is the general solution of the given system of DEs.

The constants c_1 and c_2 depend on the initial condition.