MA 138 – Calculus 2 with Life Science Applications
Linear Systems: Applications
(Section 11.2)

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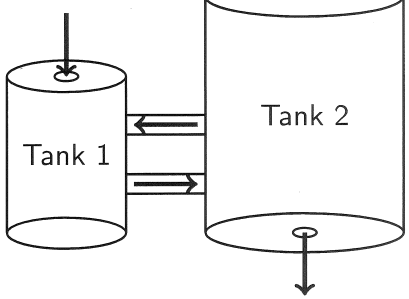
Compartment Models

- Compartment models describe flow between compartments, such as nutrient flow between lakes or the flow of a radioactive tracer between different parts of an organism.
- In the simplest situations, the resulting model is a system of linear differential equations.

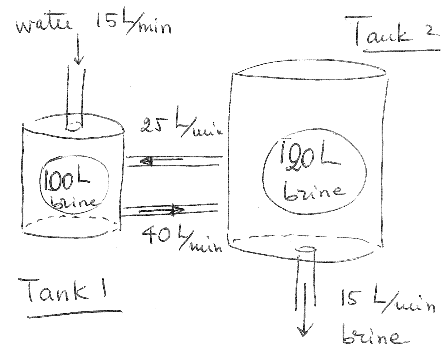
Example 1 (Online Homework #4)

Consider two brine tanks connected as shown in the figure. Pure water flows into the top of Tank 1 at a rate of 15 L/min. The brine solution is pumped from Tank 1 into Tank 2 at a rate of 40 L/min, and from Tank 2 into Tank 1 at a rate of 25 L/min. A brine solution flows out the bottom of Tank 2 at a rate of 15 L/min.



Suppose there are 100 L of brine in Tank 1 and 120 L of brine in Tank 2. Let x be the amount of salt, in kilograms, in Tank 1 after t minutes, and y the amount of salt, in kilograms, in Tank 2 after t minutes.

Assume that each tank is mixed perfectly. Set up a system of first-order differential equations that models this situation.



Let

$x(t) = x$ = amount of salt in kg in tank 1 at time t

$y(t) = y$ = amount of salt in kg in tank 2 at time t

Each tank is perfectly mixed!

$$\frac{dx}{dt} = \underbrace{-\frac{40}{100}x}_{\text{amount of salt goint into tank 2}} + \underbrace{\frac{25}{120}y}_{\text{amount of salt gained from tank 2}}$$

$$\frac{dy}{dt} = \underbrace{\frac{40}{100}x}_{\substack{\text{amount of}\\ \text{salt gained}\\ \text{from tank 1}}} - \underbrace{\frac{(25+15)}{120}y}_{\substack{\text{amount of salt}\\ \text{going into tank 1}\\ \text{and outside of}\\ \text{tank 2}}}$$

$$\therefore \begin{cases} \dfrac{dx}{dt} = -0.4x + \dfrac{5}{24}y \\[2ex] \dfrac{dy}{dt} = 0.4x - \dfrac{1}{3}y \end{cases}$$

or in Matrix form $\quad \dfrac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.4 & 5/24 \\ 0.4 & -1/3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$\det \begin{bmatrix} -0.4-\lambda & 5/24 \\ 0.4 & -1/3-\lambda \end{bmatrix} = (-0.4-\lambda)(-1/3-\lambda) - 0.4\frac{5}{24} = 0$$

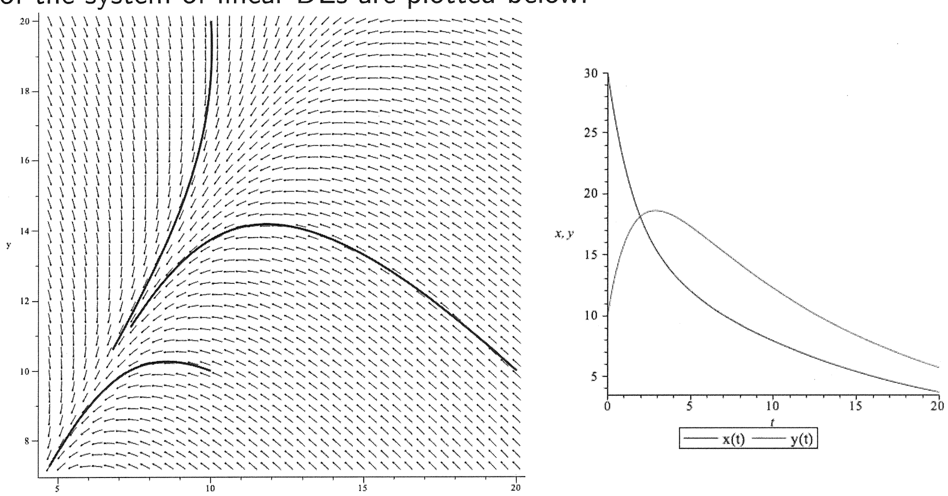
$$\Longleftrightarrow \quad \lambda^2 + \frac{2.2}{3}\lambda + \frac{1.2}{24} = 0$$

$$\Longleftrightarrow \quad 24\lambda^2 + 17.6\lambda + 1.2 = 0$$

$$\lambda_{1,2} = \frac{-17.6 \pm \sqrt{17.6^2 - 4\cdot 24 \cdot 1.2}}{48} = \frac{-17.6 \pm \sqrt{194.56}}{48} = \begin{cases} -0.076 \\ -0.6572 \end{cases}$$

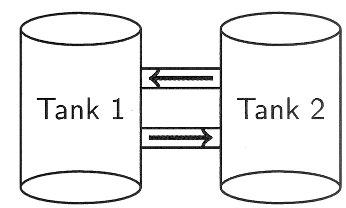
Thus $(0,0)$ is a stable equilibrium and eventually the tanks will have no salt left.

Example 1: The direction field and the graph of two particular solutions of the system of linear DEs are plotted below:



Example 2 (Online Homework #5)

Consider two brine tanks connected as shown in the figure. The brine solution is pumped from Tank 1 into Tank 2 at a rate of 10 L/min, and from Tank 2 into Tank 1 at a rate of 10 L/min. Suppose there are 50 L of brine in Tank 1 and 25 L of brine in Tank 2.



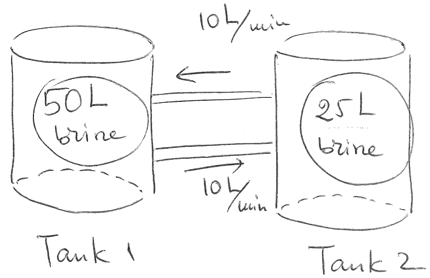
Let x be the amount of salt, in kilograms, in Tank 1 after t minutes have elapsed, and let y the amount of salt, in kilograms, in Tank 2 after t minutes have elapsed.

Assume that each tank is mixed perfectly.

If $x(0) = 7$ kg and $y(0) = 8$ kg, find the amount of salt in each tank after t minutes.

As $t \to \infty$, how much salt is in each tank?

$x(t)$ = amount of salt in kg in Tank 1 $= x$



$y(t)$ = amount of salt in kg in Tank 2 $= y$

Assume that each tank is mixed perfectly

and $x(0) = 7$ kg $\qquad y(0) = 8$ kg

Find $x(t)$ and $y(t)$.

Find x_∞ and y_∞.

$$\frac{dx}{dt} = \underbrace{-\frac{10}{50}x}_{\text{amount of salt going into Tank 2}} + \underbrace{\frac{10}{25}y}_{\text{amount of salt coming from Tank 2}}$$

$$\frac{dy}{dt} = \underbrace{\frac{10}{50}x}_{\text{amount of salt coming from Tank 1}} - \underbrace{\frac{10}{25}y}_{\text{amount of salt going into Tank 1}}$$

$\boxed{\text{Thus}}$
$$\begin{cases} \dfrac{dx}{dt} = -0.2x + 0.4y \\[2mm] \dfrac{dy}{dt} = 0.2x - 0.4y \end{cases}$$

In matrix form:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\begin{bmatrix} -0.2-\lambda & 0.4 \\ 0.2 & -0.4-\lambda \end{bmatrix} = (-0.2-\lambda)(-0.4-\lambda) - 0.08$$

$$= \cancel{0.08} + 0.2\lambda + 0.4\lambda + \lambda^2 - \cancel{0.08} =$$

$$= \lambda^2 + 0.6\lambda = 0 \implies \boxed{\lambda_1 = 0} \quad \boxed{\lambda_2 = -0.6}$$

<u>eigenvector for $\lambda_1 = 0$</u>

$$\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\iff \begin{cases} -0.2v_1 + 0.4v_2 = 0 \\ 0.2v_1 - 0.4v_2 = 0 \end{cases} \iff +v_1 - 2v_2 = 0$$

$\therefore$ $v_1 = 2v_2$ $\qquad$ choose for example $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

<u>eigenvector for $\lambda_2 = -0.6$</u>

$$\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -0.6\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \iff$$

$$\begin{cases} -0.2u_1 + 0.4u_2 = -0.6u_1 \\ 0.2u_1 - 0.4u_2 = -0.6u_2 \end{cases} \iff \begin{cases} 0.4u_1 + 0.4u_2 = 0 \\ 0.2u_1 + 0.2u_2 = 0 \end{cases}$$

$\iff$ $u_1 + u_2 = 0$ or $u_1 = -u_2$

Choose for example $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thus the general solution is :

$$\begin{bmatrix}x\\y\end{bmatrix} = c_1\begin{bmatrix}2\\1\end{bmatrix}\underbrace{e^{0\cdot t}}_{1} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{-0.6t}$$

Finally at $t=0$ $\begin{bmatrix}x(0)\\y(0)\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix}$ So

$$c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix} \qquad \underline{\underline{\text{OR}}}$$

$$\begin{bmatrix}2 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}7\\8\end{bmatrix} \Longrightarrow$$ Multiply by the inverse

$$\begin{bmatrix}c_1\\c_2\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix}\begin{bmatrix}7\\8\end{bmatrix} = \begin{bmatrix}5\\3\end{bmatrix}$$

Thus the solution to our initial value problem is given by :

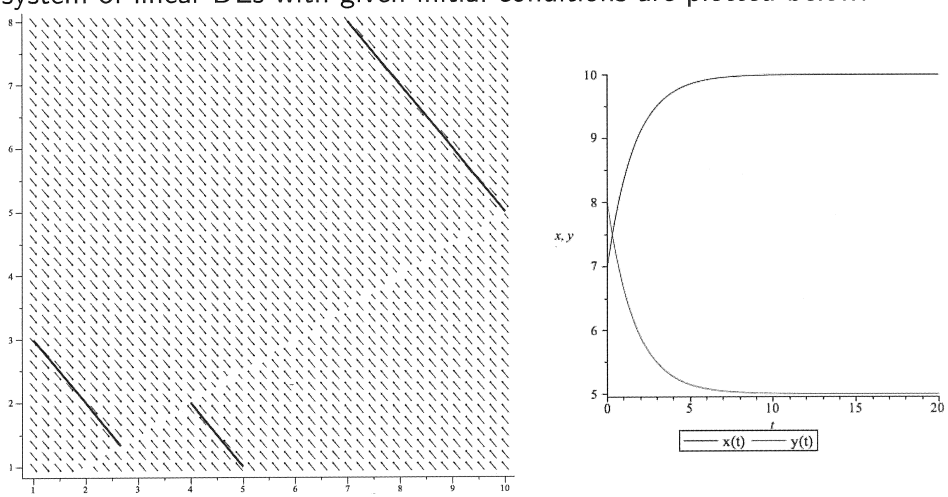
$$\begin{bmatrix}x(t)\\y(t)\end{bmatrix} = 5\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}-1\\1\end{bmatrix}e^{-0.6t}$$

$\boxed{\text{OR}}$

$$\begin{cases} x(t) = 10 - 3e^{-0.6t} \\ y(t) = 5 + 3e^{-0.6t} \end{cases}$$

As $t \to \infty$ $\boxed{x_\infty = 10}$ and $\boxed{y_\infty = 5}$

Example 2: The direction field and the graph of the two solutions of the system of linear DEs with given initial conditions are plotted below:



Higher Order Differential Equations

- (Ordinary) differential equations ($\equiv$ODEs) arise naturally in many different contexts throughout mathematics and science (social and natural). Indeed, the most accurate way of describing changes mathematically uses differentials and derivatives.
- So far we have looked only to first order differential equations.
- A simple example is Newton's Second Law of Motion, which is described by the differential equation $\quad m\dfrac{d^2x(t)}{dt^2} = F(x(t))$ (m is the constant mass of a particle subject to a force F, which depends on the position $x(t)$ of the particle at time t).
- Let F be a given function of x, y, and derivatives of y. Then an equation of the form

$$y^{(n)} = F\left(x, y, y', \cdots y^{(n-1)}\right)$$

is called an explicit ordinary differential equation of order n.

Reduction of to a First-Order System

- Differential equations can usually be solved more easily if the order of the equation can be reduced.
- Any differential equation of order n,

$$y^{(n)} = F\left(x, y, y', y'', \cdots, y^{(n-1)}\right)$$

can be written as a system of n first-order differential equations by defining a new family of unknown functions

$$y_i = y^{(i-1)}$$

for $i = 1, 2, \ldots, n$.

- Note that these new functions are related by

$$y_1' = y_2 \qquad y_2' = y_3 \quad \cdots \quad y_{n-1}' = y_n \qquad y_n' = F(x, y_1, y_2, \ldots, y_n).$$

- Your solution is then the function $y_1 = y$.

Example 3 (Online Homework #2)

Solve the following differential equation:

$$y'' - 3y' - 10y = 0$$

with the initial conditions $y = 1,\ y' = 10$ at $x = 0$.

Consider

$$\boxed{y'' - 3y' - 10y = 0 \qquad y(0) = 1 \quad y'(0) = 10}$$

that is $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} - 10y = 0$

Set $y_1 = y \qquad y_2 = y' = \dfrac{dy}{dt}$

Notice $y_1' = y' = y_2$

$$y_2' = (y')' = y'' = \text{from the original problem}$$
$$= 3y' + 10y$$
$$= 10y_1 + 3y_2$$

Thus the given equation of order 2 is equivalent to the system:

$$\begin{cases} y_1' = y_2 \\ y_2' = 10y_1 + 3y_2 \end{cases} \quad \text{OR} \quad \frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Thus: $\det\begin{bmatrix} -\lambda & 1 \\ 10 & 3-\lambda \end{bmatrix} = -\lambda(3-\lambda) - 10 = 0$

$$= \lambda^2 - 3\lambda - 10 = 0 \iff (\lambda - 5)(\lambda + 2) = 0$$

$\therefore \boxed{\lambda_1 = 5} \quad \boxed{\lambda_2 = -2}$

eigenvector for $\boxed{\lambda_1 = 5}$

$$\begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 5\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \iff \begin{cases} v_2 = 5v_1 \\ 10v_1 + 3v_2 = 5v_2 \end{cases}$$

$$\iff \begin{cases} v_2 = 5v_1 \\ 10v_1 - 2v_2 = 0 \end{cases} \iff v_2 = 5v_1$$

Choose for example $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$

eigenvector for $\boxed{\lambda_2 = -2}$

$$\begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \iff \begin{cases} u_2 = -2u_1 \\ 10u_1 + 3u_2 = -2u_2 \end{cases}$$

$$\iff u_2 = -2u_1$$

choose for example $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Thus the general solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 5 \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2t}$$

The initial condition is $y_1(0) = 1$; $y_2(0) = 10$

Thus $\begin{bmatrix} 1 \\ 10 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\boxed{\text{OR}}$ $\begin{bmatrix} 1 & 1 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix} \Longrightarrow$ multiply both sides by the inverse

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -2 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 10 \end{bmatrix} = \begin{bmatrix} 12/7 \\ -5/7 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{12}{7}\begin{bmatrix} 1 \\ 5 \end{bmatrix}e^{5t} - \frac{5}{7}\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2t}$$

or

$$\begin{cases} y_1 = \boxed{y = \frac{12}{7}e^{5t} - \frac{5}{7}e^{-2t}} \\ y_2 = y' = \frac{60}{7}e^{5t} + \frac{10}{7}e^{-2t} \end{cases}$$

notice indeed the y_2 is the derivative of $y = y_1$!!