WORKSHEET #17

#4 It is a tedious calculation to verify that

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix} = $$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and also

$$\begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = $$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\#5}$ (a)

$$\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

We solve it using Gaussian Elimination

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -5 & 1 \end{array}\right] \quad \xrightarrow{R_2-3R_1} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5-6 & 1-12 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -11 & -11 \end{array}\right] \quad \xrightarrow{-\frac{1}{11}R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c} 1 & 0 & 4-2 \\ 0 & 1 & 1 \end{array}\right] \quad \text{So} \quad \begin{array}{l} x_1 = 2 \\ x_1 = 1 \end{array}$$

(b)

$$A^{-1} = \frac{1}{-5-6}\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/11 & 2/11 \\ 3/11 & -1/11 \end{bmatrix}$$

So that

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{B}$$

$$\Longleftrightarrow \quad X = A^{-1}B = \begin{bmatrix} 5/11 & 2/11 \\ 3/11 & -1/11 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{20}{11} + \frac{2}{11} \\ 12/11 - 1/11 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\boxed{\#6}$ $\quad A = \begin{bmatrix} 5 & 2 \\ -7 & -3 \end{bmatrix}$

$$\det A = 5(-3) - 2(-7) = -15 + 14 = -1 \neq 0$$

so the inverse exists

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} -3 & -2 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -7 & -5 \end{bmatrix}$$

$\boxed{\#7}$ To find the inverse of B we need to row reduce

$$\left[\begin{array}{ccc|ccc} -1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$-R_1 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 + R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 4 & -3 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3/4 & -1/4 & 1/4 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_3 \\ R_2 + R_3 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -1/4 & 1/4 & -1/4 \\ 5/4 & 3/4 & 1/4 \\ -3/4 & -1/4 & 1/4 \end{matrix}}_{B^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$\boxed{\#8}$ To find the inverse of C we need to row reduce the matrix

$$\left[\begin{array}{ccc|ccc} 3 & 1 & -1 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

as in #7 you will see that we get

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 0 & 1/4 & 1/4 \\ 2 & -1 & -2 \\ 1 & -1/4 & -5/4 \end{matrix}}_{C^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$\boxed{\#9}$

$$A = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

So that

$$A \cdot B = \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 13 & 6 \end{bmatrix}$$

$$(A \cdot B)^{-1} = \frac{1}{6-26} \begin{bmatrix} 6 & -2 \\ -13 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6/20 & 2/20 \\ 13/20 & -1/20 \end{bmatrix}$$

Also $$A^{-1} = \frac{1}{-5} \begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3/5 & 1/5 \\ 2/5 & 1/5 \end{bmatrix}$$

$$B^{-1} = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}$$

Finally we need to compute

$$B^{-1} \cdot A^{-1} = \left(\frac{1}{4}\begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}\right)\left(-\frac{1}{5}\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}\right)$$

$$= -\frac{1}{20}\begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & -1 \end{bmatrix}$$

$$= -\frac{1}{20}\begin{bmatrix} 6 & -2 \\ -13 & 1 \end{bmatrix} = \begin{bmatrix} -6/20 & 2/20 \\ 13/20 & -1/20 \end{bmatrix}$$

$$= (AB)^{-1} \checkmark$$

$\boxed{\#10}$

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 4 & -1 \\ 8 & -1 \end{bmatrix}$$

So

$$\underbrace{\underbrace{A^{-1}A}_{I_2}B}_{B} = A^{-1}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

and $B = \begin{bmatrix} 4 & -1 \\ 8 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ -2 & 9 \end{bmatrix}$