WORKSHEET #26

#1

$$f(x,y) = \begin{bmatrix} \underbrace{x+y}_{f_1(x,y)} \\ \underbrace{x^2-y^2}_{f_2(x,y)} \end{bmatrix}$$

the Jacobi matrix is

$$Df(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2x & -2y \end{bmatrix}$$

#2

$$f(x,y) = \begin{bmatrix} e^{x-y} \\ e^{x+y} \end{bmatrix}$$

The Jacobi matrix is

$$Df(x,y) = \begin{bmatrix} e^{x-y}\cdot(1) & e^{x-y}(-1) \\ e^{x+y}(1) & e^{x+y}\cdot(1) \end{bmatrix}$$

#3

$$f(x,y) = \begin{bmatrix} 2x^2y - 3y + x \\ e^x \cdot \sin y \end{bmatrix}$$

The Jacobi matrix is

$$Df(x,y) = \begin{bmatrix} 4xy + 1 & 2x^2 - 3 \\ e^x \cdot \sin y & e^x \cos y \end{bmatrix}$$

#4 Find the linear approximation at $(1,1)$ of

$$f(x,y) = \begin{bmatrix} e^{2x-y} \\ \ln(2x-y) \end{bmatrix}$$

Notice that $f(1,1) = \begin{bmatrix} e^{2(1)-1} \\ \ln(2(1)-1) \end{bmatrix} = \begin{bmatrix} e \\ 0 \end{bmatrix}$

$$Df(x,y) = \begin{bmatrix} e^{2x-y} \cdot 2 & e^{2x-y}(-1) \\ \dfrac{1}{2x-y} \cdot 2 & \dfrac{1}{2x-y}(-1) \end{bmatrix}$$

$$Df(1,1) = \begin{bmatrix} 2e & -e \\ 2 & -1 \end{bmatrix}$$

Hence

$$L(x,y) = \begin{bmatrix} e \\ 0 \end{bmatrix} + \begin{bmatrix} 2e & -e \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix}$$

$$= \begin{bmatrix} e + 2e(x-1) - e(y-1) \\ 2(x-1) - (y-1) \end{bmatrix}$$

$\boxed{\#5}$ Find the linear approximation of

$$f(x,y) = \begin{bmatrix} e^x \sin y \\ e^{-y} \cos x \end{bmatrix} \quad \text{at} \quad (0,0)$$

Notice that $f(0,0) = \begin{bmatrix} e^0 \sin 0 \\ e^{-0} \cos(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Df(x,y) = \begin{bmatrix} e^x \sin y & e^x \cos y \\ -e^{-y} \sin x & -e^{-y} \cos x \end{bmatrix}$$

$$Df(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Hence

$$L(x,y) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

$\boxed{\#6}$ Find the linear approximation at $(-1,1)$ of $f(x,y) = \begin{bmatrix} (x+y)^2 \\ xy \end{bmatrix}$

Notice

$$f(-1,1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$Df(x,y) = \begin{bmatrix} 2(x+y)\cdot 1 & 2(x+y)(1) \\ y & x \end{bmatrix}$$

$$Df(-1,1) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

Hence

$$\mathbb{L}(x,y) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x+1 \\ y-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1 + (x+1) - (y-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ x-y+1 \end{bmatrix}$$

$\boxed{\#7}$

(a) $$\frac{d}{dt}\left[\begin{pmatrix} a \\ b \end{pmatrix} e^{kt} \right] = \begin{pmatrix} a \\ b \end{pmatrix} k \cdot e^{kt}$$

(b) $$\frac{d}{dt}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{kt} \right)$$ by part (a) is

$$k \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{kt}$$

Thus we need to solve

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\downarrow$$

$$k\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{kt} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{kt}$$

i.e. simplify $e^{kt} \neq 0$ (always)

$$\underbrace{\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\downarrow$$

direct calculation

$$\begin{bmatrix} 7-2 \\ -4-1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

i.e. $k=5$

(c) similarly to what we did in (b)

we need to solve

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

substitute $\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{kt}$

We get

$$k\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{kt} = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{kt}$$

simplify $e^{kt} \neq 0$ (always) so we need k s/t

$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = k\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

eigenvalue

$$\begin{bmatrix} 7 & -4 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

so $k = 3$