WORKSHEET #28

#1

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det\begin{bmatrix} 0-\lambda & 1 \\ 4 & 3-\lambda \end{bmatrix} = 0$$

$$\Longleftrightarrow \quad -\lambda(3-\lambda) - 4 = 0$$

$$\lambda^2 - 3\lambda - 4 = 0$$

$$(\lambda - 4)(\lambda + 1) = 0$$

So $\underline{\lambda_1 = 4}$ and $\underline{\lambda_2 = -1}$

the eigenvector corresponding to $\lambda_1 = 4$

$$\begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 4\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} v_2 = 4v_1 \\ 4v_1 + 3v_2 = 4v_2 \end{cases}$$

in either situation we obtain $\boxed{v_2 = 4v_1}$

So we can pick $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The eigenvector corresponding to $\lambda_2 = -1$

$$\begin{bmatrix} 0 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longleftrightarrow \begin{cases} v_2 = -v_1 \\ 4v_1 + 3v_2 = -v_2 \end{cases}$$

$\Longleftrightarrow$ $v_2 = -v_1$ and we can pick

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence the general solution is

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

When $t=0$ we have $x(0) = 0$ and

$y(0) = 5$ So

$$\begin{bmatrix} 0 \\ 5 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \cdot \underbrace{e^0}_{1} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \underbrace{e^0}_{1}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

Thus $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{-\frac{1}{5}\begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix}}_{\text{inverse of matrix}} \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

or $c_1 = 1$ and $c_2 = -1$

Thus $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{4t} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

$= \begin{bmatrix} e^{4t} - e^{-t} \\ 4e^{4t} + e^{-t} \end{bmatrix}$

or $x(t) = e^{4t} - e^{-t}$; $y(t) = 4e^{4t} + e^{-t}$

$\boxed{\#2}$ To solve the differential equation

$$y'' - 3y' - 10y = 0$$

We introduce $y_1(t)$ and $y_2(t)$ such that

$\boxed{y_1(t) = y(t)}$ and $\boxed{y_2(t) = y'(t)} = y_1'$

Hence $y_2' = y'' = 3y' + 10y$ or

$y_2' = 3y_2 + 10y_1$

Hence we have a system of DEs

$$y_1' = y_2$$

$$y_2' = 10y_1 + 3y_2$$

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

This is very similar to problem $\boxed{\#1}$

$\underline{\underline{y_1(0)}} = y(0) = \underline{\underline{1}}$ and $\underline{\underline{y_2(0)}} = y'(0) = \underline{\underline{10}}$

Check that $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = 5$.

Check that $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = -2$

Hence the general solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

Using the initial condition we get

$$\begin{bmatrix} 1 \\ 10 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{cases} c_1 + c_2 = 1 \\ 5c_1 - 2c_2 = 10 \end{cases}$$

Multiply the first equation by 2 and add it to the second equation

$$\begin{cases} c_1 + c_2 = 1 \\ 7c_1 + 0 \cdot c_2 = 12 \end{cases}$$

So $\underline{\underline{c_1 = 12/7}}$ and $c_2 = 1 - \frac{12}{7} = \underline{\underline{-5/7}}$

Hence $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 12/7 \begin{bmatrix} 1 \\ 5 \end{bmatrix} e^{5t} - 5/7 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$= \begin{bmatrix} 12/7\, e^{5t} - 5/7\, e^{-2t} \\ \frac{60}{7} e^{5t} + \frac{10}{7} e^{-2t} \end{bmatrix}$$

But we are interested in

$$\boxed{y_1(t) = y(t) = \frac{12}{7}e^{5t} - \frac{5}{7}e^{-2t}}$$

$\boxed{\#3}$ We the information given

$$\begin{cases} \dfrac{db}{dt} = \underline{\underline{0.06}}\, b + \underline{\underline{1}} \cdot s \\ \\ \dfrac{ds}{dt} = 0 \cdot b + \underline{\underline{0.04}}\, s \end{cases}$$

$b(0) = 500$ and $s(0) = 500$

$$\frac{d}{dt}\begin{bmatrix} b \\ s \end{bmatrix} = \begin{bmatrix} 0.06 & 1 \\ 0 & 0.04 \end{bmatrix}\begin{bmatrix} b \\ s \end{bmatrix}$$

Observe that the eigenvalues are easy to compute

$\lambda_1 = 0.06$ and $\lambda_2 = 0.04$

The corresponding eigenvectors are

$$\begin{bmatrix} 0.06 & 1 \\ 0 & 0.04 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0.06 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{cases} \cancel{0.06 V_1} + V_2 = \cancel{0.06 V_1} \\ 0.04 V_2 = 0.06 V_2 \end{cases}$$

reduces to $V_2 = 0$ Hence we can choose for example $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 0.06 & 1 \\ 0 & 0.04 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0.04 \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{cases} 0.06 V_1 + V_2 = 0.04 V_1 \\ 0.04 V_2 = 0.04 V_2 \end{cases}$$

definitely true

Hence the first equation gives us

$0.02 V_1 + V_2 = 0$ or $V_2 = -\frac{2}{100} V_1$

So we can choose for example $\begin{bmatrix} 100 \\ -2 \end{bmatrix}$

The general solution is

$$\begin{bmatrix} b(t) \\ s(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0.06t} + c_2 \begin{bmatrix} 100 \\ -2 \end{bmatrix} e^{0.04t}$$

When $t = 0$

$$\begin{bmatrix} 500 \\ 500 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 100 \\ -2 \end{bmatrix}$$

OR
$$\begin{cases} c_1 + 100\, c_2 = 500 \\ \qquad -2\, c_2 = 500 \end{cases}$$

so $c_2 = -250$ and $c_1 = 25{,}500$

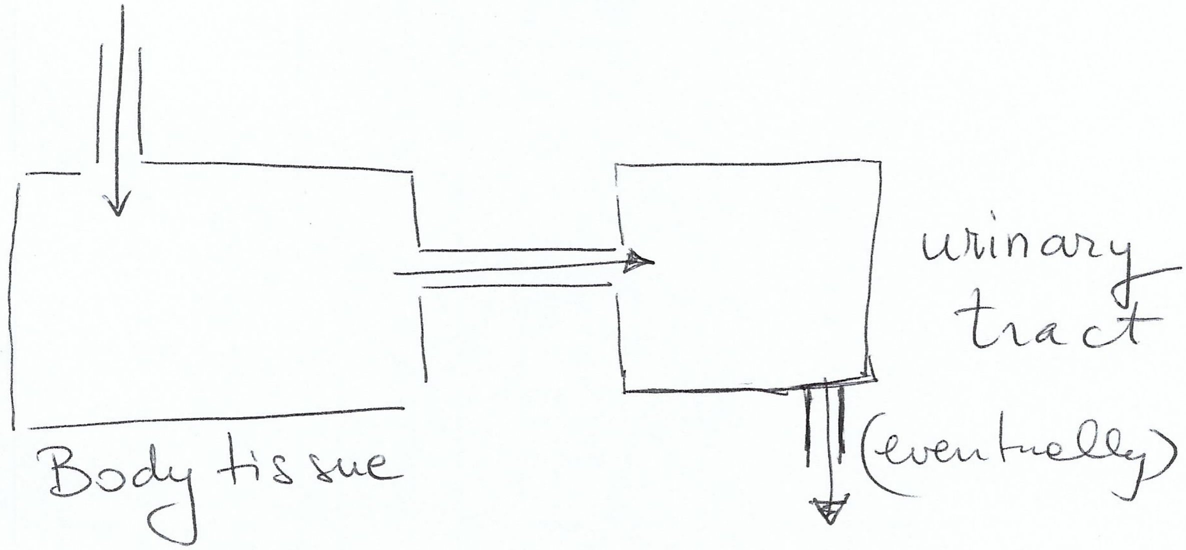
Hence

$$\begin{bmatrix} b(t) \\ s(t) \end{bmatrix} = 25{,}500 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{0.06t} - 250 \begin{bmatrix} 100 \\ -2 \end{bmatrix} e^{0.04t}$$

$$= \begin{bmatrix} \underbrace{25{,}500\, e^{0.06t} - 25000\, e^{0.04t}}_{b(t)} \\ \underbrace{500\, e^{0.04t}}_{s(t)} \end{bmatrix}$$

Let's use a compartment model

$\boxed{\#4}$



$x_1(t)$ = mg of drug in body at time t

$x_2(t)$ = mg of drug in urinary tract at time t

$$\begin{cases} \dfrac{dx_1}{dt} = -0.3\,x_1 & x_1(0) = 4 \text{ mg} \\[2ex] \dfrac{dx_2}{dt} = +0.3\,x_1 & x_2(0) = 0 \text{ mg} \end{cases}$$

$\dfrac{dx_1}{dt} = -0.3\,x_1$ is an exponential decay model so the solution is (after integration)

$$x_1(t) = \underbrace{4}_{} \cdot e^{-0.3t}$$

this is the initial condition

Hence

$$\frac{dx_2}{dt} = 0.3 \cdot (4 \cdot e^{-0.3t})$$

$$\Rightarrow \quad \int dx_2 = \int 1.2\, e^{-0.3t}\, dt$$

after substituting and separating the variables

$$x_2(t) = 1.2 \cdot \frac{1}{-0.3} e^{-0.3t} + C$$

$$= -4e^{-0.3t} + C$$

$$0 = x_2(0) = -4 + C \qquad \text{so} \quad \underline{\underline{C = 4}}$$

$$\boxed{x_2(t) = 4 - 4e^{-0.3t}}$$

notice $x_1(t) + x_2(t) = 4$. Eventually it will be removed from the body