

WORKSHEET #5

#1

$$\begin{aligned}
 \int \underbrace{x \cdot e^{-2x}}_{f g'} dx &= \underbrace{x \cdot \left[-\frac{1}{2} e^{-2x} \right]}_{f g} - \int \underbrace{1 \cdot \left(-\frac{1}{2} e^{-2x} \right)}_{f' g} dx \\
 &= -\frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx = \\
 &= -\frac{1}{2} x e^{-2x} + \frac{1}{2} \left(-\frac{1}{2} e^{-2x} \right) + C \\
 &= \boxed{-\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + C}
 \end{aligned}$$

#2

$$\begin{aligned}
 \int_1^4 x f''(x) dx &= \int_1^4 x (f'(x))' dx = \\
 &= x f'(x) \Big|_1^4 - \int_1^4 1 \cdot f'(x) dx \\
 &= x f'(x) \Big|_1^4 - \int_1^4 f'(x) dx
 \end{aligned}$$

$$\begin{aligned}
 &= x f'(x) \Big|_1^4 - f(x) \Big|_1^4 = \\
 &= 4 \cdot f'(4) - 1 \cdot f'(1) - (f(4) - f(1)) \\
 &= 4(-7) - 7 - (-2) + (-5) = \\
 &= -28 - 7 + 2 - 5 = \boxed{-38}
 \end{aligned}$$

#3 $\int \frac{4x+1}{(x-5)(x+2)} dx$

$$\frac{4x+1}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2} = \frac{A(x+2) + B(x-5)}{(x-5)(x+2)}$$

thus $4x+1 = A(x+2) + B(x-5)$

Use cover up method!

$$\begin{array}{l}
 \boxed{x=-2} \rightarrow 4(-2)+1 = A \cdot 0 + B(-2-5) \\
 \text{so } -7 = B(-7) \quad \boxed{B=1}
 \end{array}$$

$$\begin{array}{l}
 \boxed{x=5} \rightarrow 4(5)+1 = A(5+2) + B \cdot 0 \\
 21 = 7A \quad \boxed{A=3}
 \end{array}$$

Thus $\int \frac{4x+1}{x^2-3x-10} dx = \int \frac{4x+1}{(x-5)(x+2)} dx$

 $= \int \left(\frac{3}{x-5} + \frac{1}{x+2} \right) dx = 3 \ln|x-5| + \ln|x+2| + C$
 $= \boxed{\ln[|x-5|^3 \cdot |x+2|] + C}$

#4 $\int \frac{x^4+3}{x^2-4x+3} dx$ the rational function is not proper
 Let's use the long division algorithm

The diagram illustrates the long division of $x^4 + 3$ by $x^2 - 4x + 3$. The divisor $x^2 - 4x + 3$ is written vertically, and the dividend $x^4 + 3$ is written horizontally above it. The quotient is $x^2 + 4x + 13$, which is written above the division bar. The division process is shown step-by-step:

- Step 1:** Divide the leading term of the dividend by the leading term of the divisor: $x^4 / x^2 = x^2$. Write x^2 above the division bar.
- Step 2:** Multiply the entire divisor $x^2 - 4x + 3$ by x^2 and subtract the result from the dividend:

$$\begin{array}{r} x^4 + 3 \\ - (x^4 - 4x^3 + 3x^2) \\ \hline 0 \end{array}$$
 The result is $4x^3 - 3x^2 + 3$.
- Step 3:** Divide the new leading term by the divisor's leading term: $4x^3 / x^2 = 4x$. Write $4x$ above the division bar.
- Step 4:** Multiply the divisor by $4x$ and subtract:

$$\begin{array}{r} x^4 + 3 \\ - (4x^3 - 16x^2 + 12x) \\ \hline 0 \end{array}$$
 The result is $16x^2 - 12x + 3$.
- Step 5:** Divide the new leading term by the divisor's leading term: $16x^2 / x^2 = 16$. Write 16 above the division bar.
- Step 6:** Multiply the divisor by 16 and subtract:

$$\begin{array}{r} x^4 + 3 \\ - (16x^2 - 52x + 39) \\ \hline 0 \end{array}$$
 The result is $52x - 36$.

In other words

$$x^4 + 3 = (x^2 + 4x + 13)(x^2 - 4x + 3) + 40x - 36$$

Thus

$$\begin{aligned} \int \frac{x^4 + 3}{x^2 - 4x + 3} dx &= \int \frac{(x^2 + 4x + 13)(x^2 - 4x + 3) + 40x - 36}{x^2 - 4x + 3} dx \\ &= \int (x^2 + 4x + 13) dx + \int \frac{40x - 36}{(x-3)(x-1)} dx \\ &\quad \underline{\underline{\qquad\qquad\qquad}} \end{aligned}$$

$$\frac{40x - 36}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1} = \frac{A(x-1) + B(x-3)}{(x-3)(x-1)}$$

So: $40x - 36 = A(x-1) + B(x-3)$

set $x=1 \rightarrow 40 - 36 = A \cdot 0 + B(-2) \therefore B = -2$

set $x=3 \rightarrow 40 \cdot 3 - 36 = A \cdot (3-1) + B \cdot 0$

$$84 = 2A \quad \boxed{A = 42}$$

$$\frac{40x - 36}{(x-3)(x-1)} = \frac{42}{x-3} - \frac{2}{x-1}$$

Hence

$$\int \frac{x^4 + 3}{x^2 - 4x + 3} dx = \int (x^2 + 4x + 13) dx + \int \frac{42}{x-3} dx \\ - \int \frac{2}{x-1} dx$$

$$= \boxed{\frac{1}{3}x^3 + 2x^2 + 13x + 42 \ln|x-3| - 2 \ln|x-1| + C}$$