MA 138
Calculus II with Life Science Applications
FIRST MIDTERM

Spring 2023
02/07/2023

Name: Answer Key

Sect. #: ______________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ [b] [c] [d] [e]

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. [a] ■ [c] [d] [e]

2. ■ [b] [c] [d] [e]

3. [a] [b] [c] ■ [e]

4. [a] [b] ■ [d] [e]

5. [a] ■ [c] [d] [e]

6. [a] [b] [c] ■ [e]

7. [a] ■ [c] [d] [e]

8. [a] [b] [c] [d] ■

9. [a] ■ [c] [d] [e]

10. ■ [b] [c] [d] [e]

GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Lecturer	Time/Location
001–004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 110
Section #	**Recitation Instructor**	**Time/Location**
001	Kathryn Hechtel	TR 08:00 am - 08:50 am, CB 307
002	Kathryn Hechtel	TR 09:00 am - 09:50 am, CB 307
003	Davis Deaton	TR 10:00 am - 10:50 am, CB 307
004	Davis Deaton	TR 11:00 am - 11:50 am, CB 307

1. Compute the average of the function $f(x) = 3x^2 + 2$ over the interval $[1, 4]$

$$\text{avg}_f = \frac{1}{4-1}\int_1^4 (3x^2+2)\,dx$$

$$= \frac{1}{3}\left[x^3 + 2x\Big|_1^4\right] =$$

$$= \frac{1}{3}\left[(4^3 + 2\cdot 4) - (1^3 + 2\cdot 1)\right] = \frac{69}{3} = 23$$

(a) 26
$\boxed{\text{(b) } 23}$
(c) 20
(d) 17
(e) 14

2. Consider the region enclosed by the curves

$$y = x \qquad y = \frac{2}{x} - 1$$

OR $y + 1 = \frac{2}{x}$

OR $x = \frac{2}{y+1}$

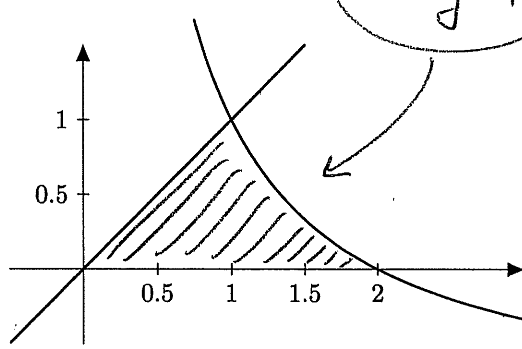
and the x-axis as indicated below.

correct

I. $\displaystyle\int_0^1 \left(\frac{2}{y+1} - y\right) dy$

II. $\displaystyle\int_0^1 \left(\frac{2}{x} - 1 - x\right) dx$

III. $\displaystyle\int_0^1 x\,dx + \int_1^2 \left(\frac{2}{x} - 1\right) dx$



Which of the above integrals measures the area of this region?

Certainly III is correct (integration wrt x)
About integration wrt y? The function below is $x = y$ and the function on top is $x = \frac{2}{y+1}$

Hence I is correct

Possibilities:

$\boxed{\text{(a) I. and III. only}}$
(b) I. only
(c) II. only
(d) III. only
(e) II. and III. only

3. Find $\int \sin(x)\cos(\cos(x))\,dx$

Set $\boxed{u = \cos(x)}$. Then $\frac{du}{dx} = -\sin(x)$

or $\boxed{-du = \sin(x)dx}$. Thus the integral becomes $\int -\cos(u)\,du$

$= -\sin(u) + C$

$= \boxed{-\sin(\cos(x)) + C}$

Possibilities:

(a) $\cos(\sin(x)) + C$

(b) $\sin(x) + \cos(x) + C$

(c) $-\cos(x)\sin(\sin(x)) + C$

(d) $-\sin(\cos(x)) + C$

(e) $\sin^2(x) + \cos^2(x) + C$

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4. Consider the following

$$\int \frac{\ln(\ln(x))}{x\ln(x)}\,dx = \int u\,du = \frac{1}{2}u^2 + C$$

What substitution did we make?

Observe that if we set $u = \ln(\ln(x))$

Then by the chain rule we get

$\frac{du}{dx} = \frac{1}{\ln(x)} \cdot \frac{1}{x}$ or $du = \frac{1}{x\ln(x)}dx$

Possibilities:

Thus (c) is the correct choice!

(a) $u = \ln(x)$

(b) $u = x\ln(x)$

(c) $u = \ln(\ln(x))$

(d) $u = 1/\ln(x)$

(e) $u = e^x$

5. Suppose that f is a differentiable function. Which of the following is equal to the integral

$$\int x^2 f'(x)\,dx$$

$$= x^2 \cdot f(x) - \int 2x \cdot f(x)\,dx$$

using integration by parts

Possibilities:

(a) $\frac{x^3}{3} f(x) + C$

(b) $\boxed{x^2 f(x) - \int 2x f(x)\,dx}$

(c) $2x f''(x) - \int x^2 f(x)\,dx$

(d) $2x f'(x) + x^2 f'(x) + C$

(e) $\frac{x^3}{3} f(x) - \int \frac{x^3}{3} f'(x)\,dx$

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6. Let f be a function defined on the interval $[1, 4]$ with continuous derivatives. Suppose in addition that we are given the following information about f and f'

$$f(1) = 4 \qquad f(4) = 6 \qquad f'(1) = -5 \qquad f'(4) = -5.$$

Find the value of

$$\int_1^4 x f''(x)\,dx$$

$$= x f'(x)\Big|_1^4 - \int_1^4 1 \cdot f'(x)\,dx =$$

use FTC!

$$= \left[4 \cdot f'(4) - f'(1)\right] - \left[f(4) - f(1)\right]$$

$$= (4(-5) - (-5)) - (6 - 4)$$

$$= -15 - 2 = -17$$

Possibilities:

(a) -2

(b) -10

(c) -15

(d) $\boxed{-17}$

(e) -25

7. Which of the following represents

$$\int \frac{4}{x^2+8x+15}\,dx?$$

Write

$$\frac{4}{x^2+8x+15} = \frac{4}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5}$$

$$= \frac{A(x+5)+B(x+3)}{(x+3)(x+5)}$$

Thus

$$4 = A(x+5)+B(x+3)$$

$\Longrightarrow$ (use cover-up)

$\underline{A=2}$ $\quad$ $\underline{B=-2}$

Possibilities:

(a) $2(\ln|x+3| + \ln|x+5|) + C$

(b) $\boxed{2(\ln|x+3| - \ln|x+5|) + C}$

(c) $2(\ln|x+5| - \ln|x+3|) + C$

(d) $4\ln(x^2+8x+15) + C$

(e) $4(-x^{-1} + 4x^2 + 15x) + C$

$$\int \frac{2}{x+3}\,dx - \int \frac{2}{x+5}\,dx$$

$$= 2\ln|x+3| - 2\ln|x+5| + C$$

8. Which of the following is the correct form of the partial fraction expansion of

$$\frac{x^2+3x-10}{(x^2+4x+6)(x^2-1)(x+1)}?$$

[**Hint:** Make sure to factor the denominator properly!]

notice that the denominator can be factored as

$$\boxed{(x^2+4x+6)(x-1)(x+1)^2}$$

and x^2+4x+6 is irreducible

So (e) is correct

(a) $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+1} + \dfrac{Dx+E}{x^2+4x+6}$

(b) $\dfrac{A}{(x+1)^2} + \dfrac{B}{x-1} + \dfrac{Cx+D}{x^2+4x+6}$

(c) $\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1} + \dfrac{D}{x^2+4x+6}$

(d) $\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-1} + \dfrac{Dx+E}{x^2+4x+6}$

(e) $\boxed{\dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1} + \dfrac{Dx+E}{x^2+4x+6}}$

9. Evaluate the following integral

$$\int_{-1}^{2} \frac{1}{x^3}\, dx$$

Observe that this is an ~~is~~ <u>improper</u> integral as the function $\frac{1}{x^3}$ becomes undefined at $x=0$.

$$\underbrace{\int_{-1}^{0} \frac{1}{x^3} dx}_{\text{diverges}} + \underbrace{\int_{0}^{2} \frac{1}{x^3} dx}_{\text{diverges}}$$

Possibilities:

(a) 3/8

$\boxed{\text{(b) Divergent}}$

(c) -1

(d) 0

(e) 1/4

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10. Let $a > 3$ be a fixed number. Evaluate the improper integral

$$\int_{a}^{\infty} \frac{1}{(x-3)^2}\, dx$$

$$= \lim_{b\to+\infty} \int_a^b \frac{1}{(x-3)^2} dx = \lim_{b\to+\infty} \left[-\frac{1}{(x-3)} \Big|_a^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{1}{b-3} - \left(-\frac{1}{a-3}\right) \right]$$

$$= \lim_{b\to+\infty} \left[\frac{1}{a-3} - \frac{1}{b-3} \right]$$

$\frac{1}{b-3} \to 0$ as $b \to +\infty$

$$= \frac{1}{a-3}$$

Possibilities:

$\boxed{\text{(a) } \frac{1}{a-3}}$

(b) 0

(c) Divergent

(d) $\frac{1}{(a+3)^2}$

(e) $\frac{1}{(a-3)^3}$

11. Suppose that a change in biomass $B(t)$ at time t during the interval $[0, 12]$ follows the equation

$$\frac{dB}{dt} = \cos\left(\frac{\pi}{6}t\right)$$

for $0 \leq t \leq 12$.

(*a*) Suppose that $B(0) = B_0$. Expresss the cumulative change in biomass during the interval $[0, t]$ as an integral.

$$B(t) - \underbrace{B(0)}_{B_0} = \int_0^t \frac{dB}{du} \cdot du = \int_0^t \cos\left(\frac{\pi}{6}u\right)du$$

(*b*) Using your previous answer, find an explicit formula for $B(t)$.

Thus

$$B(t) = B_0 + \int_0^t \cos(\pi/6\, u)\,du$$

$$= B_0 + \left(\frac{1}{\pi/6} \sin(\pi/6\, u)\Big|_0^t\right)$$

$$= \boxed{B_0 + \frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)} - \underbrace{\frac{6}{\pi}\sin(0)}_{=0}$$

(*c*) What is the value of the biomass at the end of the interval $[0, 12]$ compared with the value at time 0?

Thus $\boxed{B(t) = B_0 + \frac{6}{\pi}\sin\left(\frac{\pi}{6}t\right)}$

If $t = 12$ then $B(12) = B_0 + \underbrace{\frac{6}{\pi}\sin(2\pi)}_{=0}$

So $\boxed{B(12) = B_0 = B(0)}$ SAME

pts: /10

12. Use an appropriate substitution to evaluate the integral

$$\int x\sqrt{x+1}\,dx.$$

[Clearly indicate your answer and the steps used to arrive at that answer. Unsupported answers will receive no credit.]

Set $\boxed{u = x+1}$ so that $\frac{du}{dx} = 1$

or $\boxed{du = dx}$. Moreover $\boxed{x = u-1}$

$$\int x \cdot \sqrt{x+1}\,dx \overset{\text{becomes}}{=\!=} \int (u-1)\sqrt{u}\,du$$

$$= \int \left(u\sqrt{u} - \sqrt{u}\right) du = \int \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{1}{5/2} u^{5/2} - \frac{1}{3/2} u^{3/2} + C$$

$$= \frac{2}{5} u^{5/2} - 2/3\, u^{3/2} + C$$

$$\boxed{= \frac{2}{5}(x+1)^{5/2} - 2/3\,(x+1)^{3/2} + C}$$

OR $= \frac{2}{5}(x+1)^2\sqrt{x+1} - \frac{2}{3}(x+1)\sqrt{x+1} + C$

pts: /10

13. Use integration by parts to evaluate the integral

$$\int 9x^2 \ln(x)\, dx$$

[Clearly indicate your answer and the steps used to arrive at that answer. Unsupported answers will receive no credit.]

$$\int \underbrace{9x^2}_{g'} \cdot \underbrace{\ln(x)}_{f} \, dx =$$

$$= \underbrace{3x^3}_{g} \underbrace{\ln(x)}_{f} - \int \underbrace{3x^3}_{g} \cdot \underbrace{\frac{1}{x}}_{f'} \, dx$$

$$= 3x^3 \ln(x) - \int 3x^2 \, dx$$

$$\boxed{= 3x^3 \ln(x) - x^3 + C}$$

$$= x^3 \left[-1 + 3\ln(x) \right] + C$$

pts: /10

14. Evaluate the following integral

$$\int \frac{4x^2 - x - 1}{(x+1)^2(x-3)}\, dx$$

[Clearly indicate your answer and the steps used to arrive at that answer. Unsupported answers will receive no credit.]

We seek for a decomposition of the form

$$\frac{4x^2-x-1}{(x+1)^2(x-3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-3}$$

$$= \frac{A(x+1)(x-3) + B(x-3) + C(x+1)^2}{(x+1)^2(x-3)}$$

Thus

(*) $\boxed{4x^2-x-1 = A(x+1)(x-3) + B(x-3) + C(x+1)^2}$

If we set $\boxed{x=-1}$ in (*) we get $B=-1$

If we set $\boxed{x=3}$ in (*) we get $C=2$

Finally, if we set $x=0$ (for example) in *

we get $A=2$ Thus

$$\int \frac{4x^2-x-1}{(x+1)^2(x-3)}\,dx = \int \left(\frac{2}{x+1} - \frac{1}{(x+1)^2} + \frac{2}{x-3} \right) dx$$

$$= \boxed{2\ln|x+1| + \frac{1}{x+1} + 2\ln|x-3| + C}$$

OR $= \ln\left[\left((x+1)(x-3)\right)^2\right] + \frac{1}{x+1} + C$

pts: /10

15. Integrate the following improper integrals:

[Clearly indicate your answer and the steps used to arrive at that answer. Unsupported answers will receive no credit.]

(a) $\displaystyle\int_3^6 \frac{1}{x-3}\,dx$

use the substitution $u = x-3$. Thus $du = dx$. The integral becomes

$$\int_0^3 \frac{1}{u}\,du = \lim_{a\to 0^+} \int_a^3 \frac{1}{u}\,du =$$

$$= \lim_{a\to 0^+}\left[\ln|u|\Big|_a^3\right] = \lim_{a\to 0^+}\left[\ln(3) - \ln(a)\right]$$

$$= \ln(3) - (-\infty) = +\infty$$

diverges

(b) $\displaystyle\int_1^\infty 3x^2 e^{-x^3}\,dx$

Using the substitution $u = x^3$

$\dfrac{du}{dx} = 3x^2$ or $du = 3x^2dx$, the integral becomes $\displaystyle\int_1^{+\infty} e^{-u}\,du$

$$= \lim_{b\to\infty}\int_1^b e^{-u}\,du = \lim_{b\to\infty} -e^{-u}\Big|_1^b$$

converges

$$= \lim_{b\to\infty}\left[\underbrace{-e^{-b}}_{\to 0 \text{ as } b\to+\infty} + e^{-1}\right] = \boxed{\frac{1}{e}} \approxeq 0.3678$$

pts: /10

Bonus. (*a*) Explain why the inequalities $0 \le \dfrac{x^3}{\sqrt{x^{12}+1}} < \dfrac{1}{x^3}$ hold for any $x \ge 1$.

Observe that $x^{12} < x^{12}+1$. The $\sqrt{\cdot}$ is an increasing function so $\sqrt{x^{12}} < \sqrt{x^{12}+1}$. So $x^6 < \sqrt{x^{12}+1}$. Passing to the reciprocals $\dfrac{1}{\sqrt{x^{12}+1}} < \dfrac{1}{x^6}$. Multiply by x^3 to obtain

(*b*) Use part (*a*) and the Comparison Test for inproper integrals to determine whether or not the following integral

$$\int_1^\infty \frac{x^3}{\sqrt{x^{12}+1}}\,dx$$

converges. Explain.

By Part (a)

$$0 \le \int_1^\infty \frac{x^3}{\sqrt{x^{12}+1}}\,dx \le \int_1^\infty \frac{1}{x^3}\,dx$$

$$= \lim_{b\to\infty} \int_1^b x^{-3}\,dx = \lim_{b\to\infty} -\frac{1}{2}x^{-2}\Big|_1^b$$

$$= \lim_{b\to\infty}\Big(\underbrace{-\frac{1}{2b^2}}_{\to 0} + \frac{1}{2}\Big) = 1/2$$

Hence $\int_1^\infty \dfrac{x^3}{\sqrt{x^{12}+1}}\,dx$ converges

pts: /10