MA 138
Calculus II with Life Science Applications
FINAL EXAM

Spring 2024
05/01/2024

Name: Answer Key
Sect. #: ____________

Do not remove this answer page — you will return the whole exam. No books or notes may be used. Use the backs of the question papers for scratch paper. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The **first part of the exam** consists of 10 multiple choice questions, each worth 5 points. Record your answers on this page by filling in the box corresponding to the correct answer. For example, if (a) is correct, you must write

■ b c d e

Do not circle answers on this page, but please do circle the letter of each correct response in the body of the exam. It is your responsibility to make it CLEAR which response has been chosen. You will not get credit unless the correct answer has been marked on **both** this page **and** in the body of the exam.

The **second part of the exam** consists of five open-response questions and one bonus question. When answering these questions, check your answers when possible. Clearly indicate your answer and the reasoning used to arrive at that answer. *Unsupported answers may receive NO credit.*

1. ■ b c d e
2. a b ■ d e
3. a b c d ■
4. ■ b c d e
5. a b ■ d e
6. a b ■ d e
7. a b c d ■
8. ■ b c d e
9. a b c ■ e
10. a ■ c d e

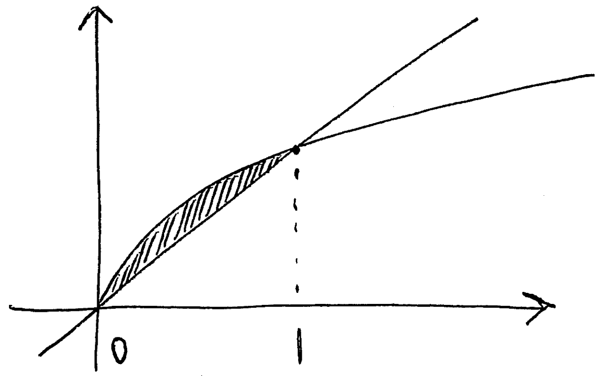
GOOD LUCK!

QUESTION	SCORE	OUT OF
Multiple Choice		50 pts
11.		10 pts
12.		10 pts
13.		10 pts
14.		10 pts
15.		10 pts
Bonus.		10 pts
TOTAL		100 pts

Please make sure to list the correct section number on the front page of your exam. In case you forgot your section number, consult the following table:

Sections #	Lecturer	Time/Location
001-004	Alberto Corso	MWF 10:00 am - 10:50 am, CB 212
Section #	**Recitation Instructor**	**Time/Location**
001	Ian Robinson	TR 09:00 am - 09:50 am, CB 307
002	Ian Robinson	TR 10:00 am - 10:50 am, CB 307

1. The area of the region enclosed by the curves $y = \sqrt{x}$ and $y = x$ is described by one of the following integrals. (**Hint:** Draw a picture of the two curves and the area they enclose).



the intersection pts are

$$\sqrt{x} = y = x$$

so $(\sqrt{x})^2 = x^2$

$x = x^2$ or

$x(x-1) = 0$

Possibilities:

(a) $\int_0^1 [\sqrt{x} - x]\, dx$

(b) $\int_0^1 [x\sqrt{x}]\, dx$

(c) $\int_0^1 [x - \sqrt{x}]\, dx$

(d) $\int_0^1 \sqrt{x}\, dx$

(e) None of the above

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2. Evaluate the improper integral

$$\int_e^\infty \frac{1}{x(\ln(x))^4}\, dx$$

Set $u = \ln x$ $\quad du = \frac{1}{x} dx$. Thus the integral becomes

$$\int_1^\infty \frac{1}{u^4}\, du = \lim_{b \to \infty} \int_1^b u^{-4}\, du$$

$$= \lim_{b \to \infty} -\frac{1}{3} u^{-3} \Big|_1^b = \lim_{b \to \infty} \left[\frac{1}{3} - \underbrace{\frac{1}{3b^3}}_{\to 0} \right]$$

$$= 1/3$$

(a) The integral diverges

(b) $\frac{1}{4}$

(c) $\frac{1}{3}$

(d) $\frac{1}{2}$

(e) 1

3. The augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

can be put into the row-reduced form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

using just two row operations. Which row operations will accomplish this?

(**Note:** R_i denotes the entries on the i-th row)

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 7 \end{array}\right] \rightsquigarrow \underset{\text{produces}}{R_2 - R_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\rightsquigarrow \underset{\text{produces}}{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Possibilities:

(a) First $R_2 - R_3$, then $R_1 - 2R_2$

(b) First $R_1 + 2R_2$, then $R_2 - R_3$

(c) First $\frac{1}{7}R_3$, then $\frac{1}{3}R_2$

(d) First $R_3 - R_2$, then $R_2 + 2R_1$

$\boxed{\text{(e) First } R_2 - R_3, \text{ then } R_1 + 2R_2}$

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4. The eigenvalues of the matrix $\begin{bmatrix} 0 & 4 \\ 3 & 1 \end{bmatrix}$ are:

$$\det \begin{bmatrix} 0-\lambda & 4 \\ 3 & 1-\lambda \end{bmatrix} = 0 \iff$$

$$-\lambda(1-\lambda) - 12 = 0 \iff$$

$$\lambda^2 - \lambda - 12 = 0 \iff$$

$$(\lambda - 4)(\lambda + 3) = 0 \quad \therefore \lambda_1 = 4, \ \lambda_2 = -3$$

$\boxed{\text{(a) } 4 \text{ and } -3}$

(b) 4 and 3

(c) -4 and -3

(d) ± 4 and ± 3

(e) 0 and 1

5. There is a correlation between the amount of oxygen dissolved in a body of water and the depth of the water: oxygen is most abundant near the surface with lesser amounts at deeper levels. Suppose that the amount of dissolved oxygen is measured at several depths in a lake and the following data are collected:

depth (ft)	dissolved oxygen (mg/L)
2	10.5
10	6.0
20	0.5

We expect a linear relationship between the depth (d) and the dissolved oxygen (O_2), that is: $\boxed{O_2 = md + b}$ for appropriate values of m and b. Which of the following systems should we solve in order to find the least-squares solution to this linear problem?

Possibilities:

(a) $\begin{bmatrix} 3 & 17 \\ 17 & 146.5 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 32 \\ 91 \end{bmatrix}$

(b) $\begin{bmatrix} 146.5 & 17 \\ 17 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 32 \end{bmatrix}$

(c) $\begin{bmatrix} 504 & 32 \\ 32 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 17 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 32 \\ 32 & 504 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 17 \\ 91 \end{bmatrix}$

(e) $\begin{bmatrix} 5 & 21 & 41 \\ 21 & 101 & 201 \\ 41 & 201 & 401 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 17 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 10 & 1 \\ 20 & 1 \end{bmatrix}}_{A}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 10.5 \\ 6.0 \\ 0.5 \end{bmatrix}$$

Multiply by A^T on both sides to obtain

$$\begin{bmatrix} 504 & 32 \\ 32 & 3 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 91 \\ 17 \end{bmatrix}$$

6. What is the linearization at the point $(1, 2)$ of the vector-valued function:

$$\mathbf{f}(x, y) = \begin{bmatrix} x^2y \\ x + y \end{bmatrix}$$

$$f(1,2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Possibilities:

(a) $\mathbf{L}(x, y) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$

(b) $\mathbf{L}(x, y) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$

(c) $\mathbf{L}(x, y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$

(d) $\mathbf{L}(x, y) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$

(e) $\mathbf{L}(x, y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$

$$Df(x,y) = \begin{bmatrix} 2xy & x^2 \\ 1 & 1 \end{bmatrix}$$

$$Df(1,2) = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\therefore L(x,y) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

7. Which of the following is the general solution to the system of differential equations.

$$\frac{dx}{dt} = x + 4y$$
$$\frac{dy}{dt} = 4x + y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

To find the eigenvalues

$$\det\begin{bmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{bmatrix} = 0$$

$$\iff (1-\lambda)^2 - 16 = 0 \quad \text{or}$$

$$(1-\lambda) = \pm\sqrt{16} \qquad \lambda_{1,2} = 1 \pm 4$$

$$\lambda_1 = 5 \qquad \lambda_2 = -3$$

eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Possibilities:

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{c_1 t} - 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{c_2 t}$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} 5e^t + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} - 3e^t$

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$

(e) $\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t}}$

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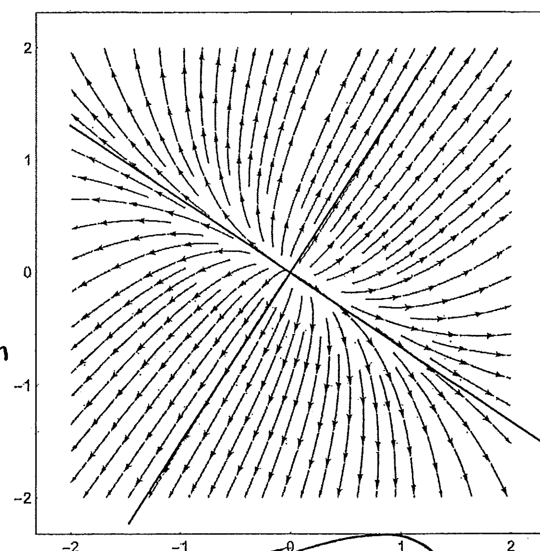
8. Consider a system of linear autonomous differential equations of the form $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and A is a 2×2 matrix with constant entries.

The direction field of the above system is shown on the right-hand side, and the lines through the eigenvectors of A are graphed as well.



Which of the following could be the characteristic polynomial of the matrix A?

(**Hint:** analyze the eigenvalues for each polynomial.)

We need to find the equation that has two positive real eigenvalues.

Possibilities:

(a) $\boxed{\lambda^2 - 5\lambda + 5}$ $\quad \lambda_{1,2} = \frac{5 \pm \sqrt{25-20}}{2} = \frac{5 \pm \sqrt{5}}{2}$ both positive

(b) $\lambda^2 - 3\lambda + 3$ $\longrightarrow$ complex eigenvalues

(c) $\lambda^2 - 4\lambda + 10$ $\longrightarrow$ Complex eigenvalues

(d) $\lambda^2 - 3\lambda - 4$ $\longrightarrow$ $\lambda_1 = 4$ & $\lambda_2 = -1$

(e) $\lambda^2 + 3\lambda + 20$ $\longrightarrow$ Complex eigenvalues

9. What type of equilibrium point is $(0,0)$ for this linear system of differential equations?

$$\frac{dx}{dt} = x - y \qquad \frac{dy}{dt} = x + y$$

$$\frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Possibilities:

(a) It is an unstable node, a source.

(b) It is a stable node, a sink.

(c) It is a saddle point.

(d) It is an unstable spiral.

(e) It is a stable spiral.

the eigenvalues are

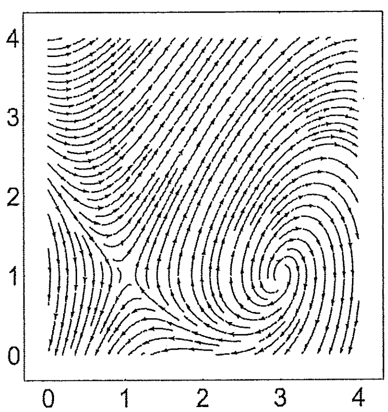
$$\det\begin{bmatrix}1-\lambda & -1\\1 & 1-\lambda\end{bmatrix} = 0$$

$$\Leftrightarrow (1-\lambda)^2 + 1 = 0 \qquad \boxed{\lambda_{1,2} = 1 \pm i}$$

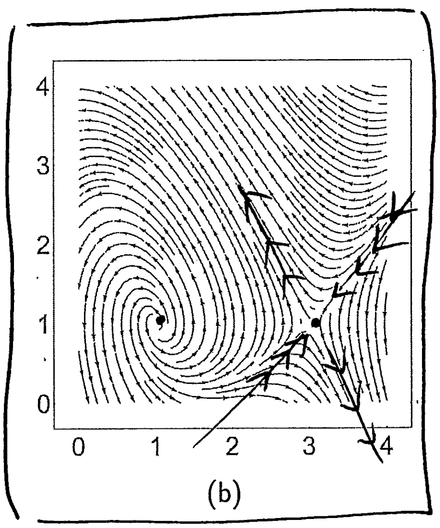
10. The points $(1,1)$ and $(3,1)$ as equilibria for the nonlinear system of differential equations:

$$\frac{dx}{dt} = 1 - y \qquad \frac{dy}{dt} = y - (x-2)^2.$$

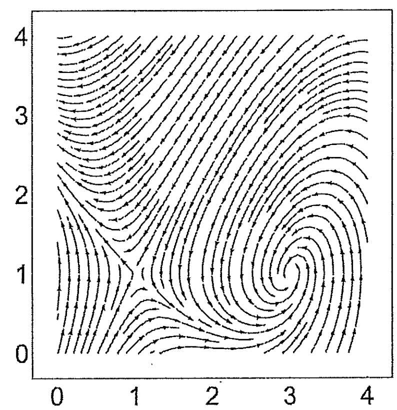
Choose the correct direction field for the given nonlinear system of differential equations.



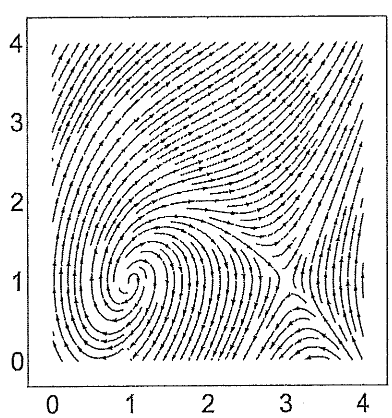
(a)



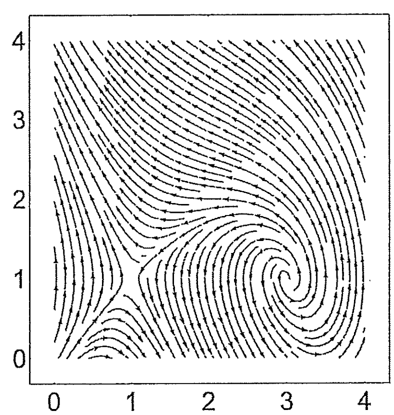
(b)



(c)



(d)



(e)

To find the equilibria we need to solve

$$\frac{dx}{dt} = 0 = \frac{dy}{dt} \quad \text{OR} \quad \begin{cases} 1 - y = 0 \\ y - (x-2)^2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} y = 1 \\ (x-2)^2 = 1 \end{cases} \Rightarrow \quad x = 2 \pm 1 \begin{cases} 1 \\ 3 \end{cases}$$

Thus $\boxed{\hat{P}_1 = (1,1)}$ and $\boxed{\hat{P}_2 = (3,1)}$

The Jacobi matrix is $\begin{bmatrix} 0 & -1 \\ -2(x-2) & 1 \end{bmatrix}$

Thus the Jacobi matrix at $\boxed{(1,1)}$ is

$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$. The eigenvalues are obtained from $(0-\lambda)(1-\lambda) + 2 = 0 \iff$

$\lambda^2 - \lambda + 2 = 0$ OR $\lambda_{1,2} \frac{1 \pm \sqrt{1-8}}{2}$

complex eigenvalues with positive real part

This leaves two choices: (b) OR (d)

the Jacobi matrix at $\boxed{(3,1)}$ is

$\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}$. The eigenvalues are

$-\lambda(1-\lambda) - 2 = 0 \iff \lambda^2 - \lambda - 2 = 0$

$\iff (\lambda - 2)(\lambda + 1) = 0$ $\quad \lambda_1 = 2$ $\quad \lambda_2 = -1$

The eigenvector corresponding to $\boxed{\lambda_1 = 2}$ satisfies
$$\begin{bmatrix} 0 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 2\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{OR}$$

$$\begin{cases} -u_2 = 2u_1 \\ -2u_1 + u_2 = 2u_2 \end{cases} \iff u_2 = -2u_1$$

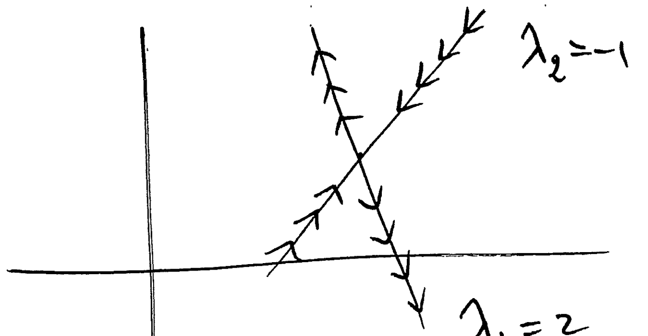
Hence we can choose for example $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

The eigenvector corresponding to $\boxed{\lambda_2 = -1}$ satisfies
$$\begin{bmatrix} 0 & -1 \\ -2 & +1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{OR}$$

$$\begin{cases} -v_2 = -v_1 \\ -2v_1 + v_2 = -v_2 \end{cases} \iff v_1 = v_2$$

Hence we can choose $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus the trajectories at the equilibrium $(3,1)$ along the eigenvectors are



This corresponds to $\boxed{(b)}$

11. Separate variables and use partial fractions to solve the following differential equation

$$\frac{dy}{dx} = \frac{-2y}{x(x-2)}$$

with initial condition $y(4) = 8$.

Separating the variables gives us: $\frac{1}{y}\,dy = \frac{-2}{x(x-2)}\,dx$

Thus $\int \frac{1}{y}\,dy = \int \frac{-2}{x(x-2)}\,dx$. We use partial fractions for the right-hand side $\frac{-2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$

$\Longleftrightarrow$ $-2 = A(x-2) + Bx$. Evaluating at $x = 0$ and $x = 2$ we obtain

$-2 = A(0-2) + B\cdot 0 \Longleftrightarrow \boxed{A=1}$

$-2 = A(2-2) + B\cdot 2 \Longleftrightarrow \boxed{B=-1}$

Thus $\frac{-2}{x(x-2)} = \frac{1}{x} - \frac{1}{x-2}$

$\int \frac{1}{y}\,dy = \int \left(\frac{1}{x} - \frac{1}{x-2}\right) dx \Longleftrightarrow \ln y = \ln x - \ln(x-2) + C$

$\Longleftrightarrow \ln y = \ln\left(\frac{x}{x-2}\right) + C$ Take exp of both sides

$y = A\cdot\frac{x}{x-2}$, when $A = e^{C}$. Using the initial condition $y(4)=8$ we obtain

$8 = A\cdot\frac{4}{4-2}$ OR $A=4$

Thus $\boxed{y = \frac{4x}{x-2}}$

pts: /10

$$\boxed{p = 18.838 + 19.089 \ln w} \qquad p(100) = 106.746$$

12. A healthy child's systolic blood pressure p (in millimeters of mercury) and weight w (in pounds) are approximately related by the equation

$$p = a + b \cdot \ln w,$$

where a and b are constants. Use the following experimental data

w	44	81	131
$\ln w$	3.78	4.41	4.88
p	91	103	112

to estimate the systolic blood pressure of a healthy child weighing 100 pounds.

(*a*) Set up a system of three linear equations that satisfy the above relationship. Solve your system to find the *least square solution*: $p = \hat{a} + \hat{b} \cdot \ln w$.

$$91 = a + b \cdot 3.78$$
$$103 = a + b \cdot 4.41$$
$$112 = a + b \cdot 4.88$$

OR

$$\underbrace{\begin{bmatrix} 1 & 3.78 \\ 1 & 4.41 \\ 1 & 4.88 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 91 \\ 103 \\ 112 \end{bmatrix}}_{B}$$

We need to multiply the above equation by A^T

$$\begin{bmatrix} 1 & 1 & 1 \\ 3.78 & 4.41 & 4.88 \end{bmatrix} \begin{bmatrix} 1 & 3.78 \\ 1 & 4.41 \\ 1 & 4.88 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3.78 & 4.41 & 4.88 \end{bmatrix} \begin{bmatrix} 91 \\ 103 \\ 112 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 3 & 13.07 \\ 13.07 & 57.5509 \end{bmatrix}}_{A^TA} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 306 \\ 1{,}344.77 \end{bmatrix}$$

Now multiply both sides by $(A^TA)^{-1}$ to get:

$$\begin{bmatrix} \hat{a} \\ \hat{b} \end{bmatrix} = \frac{1}{1.8278} \begin{bmatrix} 57.5509 & -13.07 \\ -13.07 & 3 \end{bmatrix} \begin{bmatrix} 306 \\ 1{,}344.77 \end{bmatrix} = \begin{bmatrix} 18.838 \\ 19.089 \end{bmatrix}$$

(*b*) Using the least square solution found in (*a*), estimate the systolic pressure of a child weighing $w = 100$ pounds.

$p = 18.838 + 19.089 \ln w$ when $w = 100$

we get $\boxed{p(100) = 106.746}$

pts: /10

13. What is the linearization at the point $(0,\pi)$ of the vector-valued function

$$\mathbf{f}(x,y) = \begin{bmatrix} 2x+4y \\ \sin(x+2y) \end{bmatrix}$$

Recall

$$\mathbf{L}(x,y) = \mathbf{f}(x_0,y_0) + D\mathbf{f}(x_0,y_0)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

Now, $\mathbf{f}(0,\pi) = \begin{bmatrix} 2\cdot 0 + 4\pi \\ \sin(0+2\pi) \end{bmatrix} = \begin{bmatrix} 4\pi \\ 0 \end{bmatrix}$

$$D\mathbf{f}(x,y) = \begin{bmatrix} 2 & 4 \\ \cos(x+2y)\cdot 1 & \cos(x+2y)\cdot 2 \end{bmatrix}$$

$$D\mathbf{f}(0,\pi) = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

Thus $\boxed{\mathbf{L}(x,y) = \begin{bmatrix} 4\pi \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y-\pi \end{bmatrix}}$

$$= \begin{bmatrix} 4\pi + 2x + 4(y-\pi) \\ 0 + x + 2(y-\pi) \end{bmatrix} = \begin{bmatrix} 2x+4y \\ x+2y-2\pi \end{bmatrix}$$

pts: /10

14. Consider the system of linear autonomous differential equations below:

$$\frac{dx}{dt} = 12x - 2y$$
$$\frac{dy}{dt} = -2x + 9y.$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 12 & -2 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

(a) Find the general solution of the above system.

We need the eigenvalues of the matrix $\det\begin{bmatrix} 12-\lambda & -2 \\ -2 & 9-\lambda \end{bmatrix} = 0$

$\iff (12-\lambda)(9-\lambda) - 4 = 0 \iff \lambda^2 - 21\lambda + 104 = 0$

$\iff (\lambda - 8)(\lambda - 13) = 0 \quad \therefore \underline{\underline{\lambda_1 = 8}} \quad \underline{\underline{\lambda_2 = 13}}$

Eigenvectors:

$\lambda_1 = 8 \quad \begin{bmatrix} 12 & -2 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 8\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

$\iff 4u_1 - 2u_2 = 0 \iff u_2 = 2u_1$

$\lambda_2 = 13 \quad \begin{bmatrix} 12 & -2 \\ -2 & 9 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 13\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$\iff -v_1 - 2v_2 = 0$

$\iff v_1 = -2v_2$

$$\boxed{\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{8t} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{13t}}$$

(b) Find the particular solution of this system such that $x(0) = 15$ and $y(0) = -10$.

$$\begin{bmatrix} 15 \\ -10 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{8\cdot 0} + c_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{13\cdot 0} \iff \begin{bmatrix} +15 \\ -10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$\underline{\text{i.e.}} \begin{cases} c_1 - 2c_2 = 15 \\ 2c_1 + c_2 = -10 \end{cases} \iff \boxed{\begin{matrix} c_1 = -1 \\ c_2 = -8 \end{matrix}}$

$$\boxed{\begin{matrix} x(t) = -e^{8t} + 16e^{13t} \\ y(t) = -2e^{8t} - 8e^{13t} \end{matrix}}$$

(c) Classify the stability of the equilibrium point $(0,0)$. Justify your answer.

$(0,0)$ is an unstable node (source) since both eigenvalues are positive

pts: /10

15. Consider the system of nonlinear autonomous differential equations below:

$$\frac{dx_1}{dt} = (x_2 - x_1)(x_2 - 2)$$
$$\frac{dx_2}{dt} = x_2(x_1 - 1).$$

(*a*) This system could be written in the form $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function. What is the Jacobian matrix of this $\mathbf{f}$?

$f_1 = x_2^2 - 2x_2 - x_1x_2 + 2x_1$

$f_2 = x_1x_2 - x_2$ Jacobi matrix $(Df)(x_1,x_2) = \begin{bmatrix} 2-x_2 & 2x_2-2-x_1 \\ x_2 & x_1-1 \end{bmatrix}$

(*b*) This system has three equilibrium points. One of them is the point $(0,0)$, which is a saddle point. Find the other two equilibrium points of this system, and use the Hartman-Grobman Theorem to classify them (the classifications are source, sink, saddle point, stable spiral, and unstable spiral).

We need to solve $\begin{cases} (x_2-x_1)(x_2-2)=0 \\ x_2(x_1-1)=0 \end{cases}$

Since we want $x_2 \neq 0$ we get $x_1 = 1$ from the second equation. Hence the first equation becomes

$(x_2-1)(x_2-2)=0$ so that $x_2 = 1$ OR $x_2 = 2$

Thus the 2 equilibria are $\boxed{(1,1)}$ and $\boxed{(1,2)}$

Jacobi matrix for $(1,1)$: $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $(1-\lambda)(-\lambda)+1=0$ $\lambda^2-\lambda+1=0$

$\lambda_{1,2} = \frac{1 \pm \sqrt{1-4}}{2}$ complex with positive real part

Jacobi matrix for $(1,2)$: $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ has eigenvalues $(-\lambda)^2 - 2 = 0$ OR $\lambda^2 = 2$

$\lambda_{1,2} = \pm\sqrt{2}$ real opposite sign

The equilibrium point $(1,1)$ is a unstable spiral

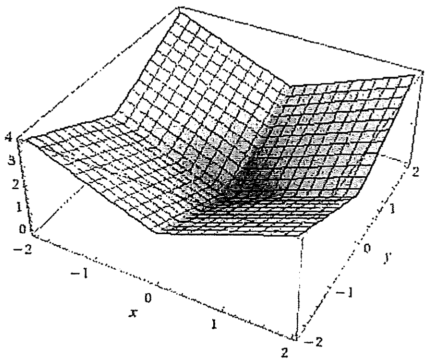
The equilibrium point $(1,2)$ is a saddle point

pts: /10

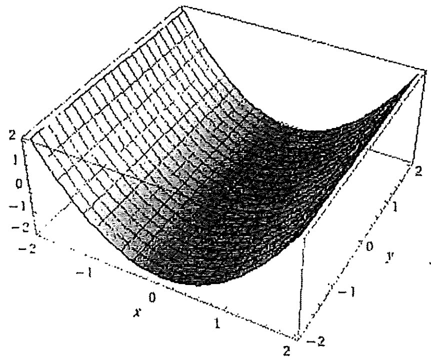
Bonus. Match each of the following functions of two variables x and y

$$f(x,y) = (x^2 - y^2)^2 \qquad g(x,y) = |x| + |y| \qquad h(x,y) = -xy\,e^{-x^2-y^2} \qquad k(x,y) = x^2 - 2$$

with its graph (labeled A.-D.) and its level curves (labeled I.-IV.).

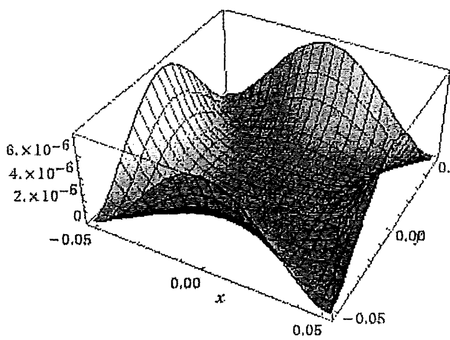


A.

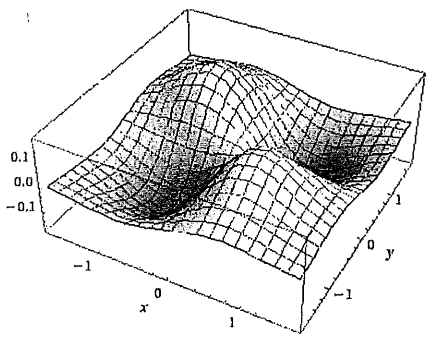


B.

$f(x,y)$ corresponds to graph C and level curves II

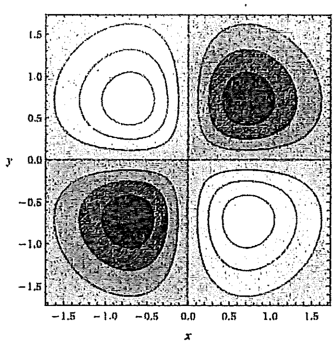


C.

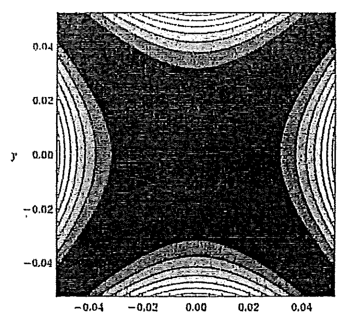


D.

$g(x,y)$ corresponds to graph A and level curves IV

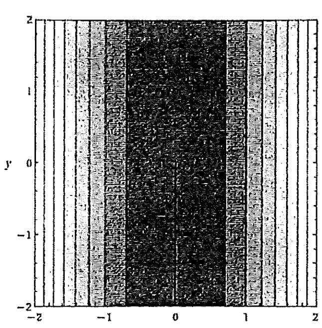


I.

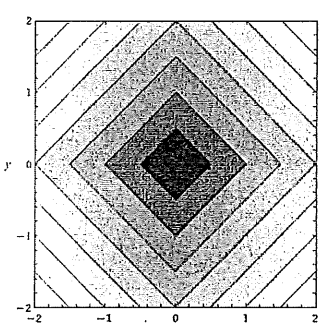


II.

$h(x,y)$ corresponds to graph D and level curves I



III.



IV.

$k(x,y)$ corresponds to graph B and level curves III

pts: /10