MA 162 : Finite Mathematics - Chapter 3

Matrix Algebra

University of Kentucky

3.2 Introduction to Matrices

A **matrix** is a rectangular array of numbers. Matrices are useful in organizing and manipulating large amounts of data.

$$\begin{array}{rc} \begin{array}{r} \textit{Row 1} \longrightarrow \\ \textit{Row 2} \longrightarrow \end{array} & \begin{bmatrix} 5 & 8 & 1 \\ 2 & 4 & 0 \end{bmatrix} \\ & \begin{array}{ccc} \textit{Column 1} & \textit{Column 2} & \textit{Column 3} \end{array} \end{array}$$

(1,3) entry (or element)

row 1 column 3

2 by 3

The **dimension** of this matrix is 2×3 since there are 2 rows and 3 columns.

3.2 Introduction to Matrices

A matrix that has the same number of rows as columns is called a **square** matrix.

$$\begin{bmatrix} 2 & 8 & 1 \\ 4 & 7 & 6 \\ 1 & 0 & 9 \end{bmatrix}$$

A matrix consisting of all zeros is called a **zero** matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A square matrix which has 1's along the main diagonal and zeros elsewhere is called an **identity** matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.2 Introduction to Matrices

A matrix that has only one row is called a **row matrix** or **row vector**

$$\begin{bmatrix} 2 & 8 & 1 & 7 \end{bmatrix}$$

A matrix that has only one column is called a **column matrix** or **column vector**

$$\begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$$

Two matrices are **equal** if they have the same size and each corresponding entry is equal.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

3.2 Matrix Addition and Subtraction

If two matrices are the same dimension, they can be added or subtracted.

Example 1: Let $A = \begin{bmatrix} -4 & 1 & 9 \\ 3 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & -2 \\ 1 & 0 & 6 \end{bmatrix}$.

- Find $A + B$

$$= \begin{bmatrix} -4+7 & 1+3 & 9-2 \\ 3+1 & -5+0 & 2+6 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 7 \\ 4 & -5 & 8 \end{bmatrix}$$

- Find $B - A$

$$= \begin{bmatrix} 7-(-4) & 3-1 & -2-9 \\ 1-3 & 0-(-5) & 6-2 \end{bmatrix} = \begin{bmatrix} 11 & 2 & -11 \\ -2 & 5 & 4 \end{bmatrix}$$

3.2 Scalar Multiplication

When we multiply a matrix by a scalar, we multiply each entry by that scalar

Example 2: Let $A = \begin{bmatrix} -4 & 1 & 9 \\ 3 & -5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & -2 \\ 1 & 0 & 6 \end{bmatrix}$.

- Find $2A$

$$2\begin{bmatrix} -4 & 1 & 9 \\ 3 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -8 & 2 & 18 \\ 6 & -10 & 4 \end{bmatrix}$$

- Find $3B - 2A$

$$= \underbrace{\begin{bmatrix} 21 & 9 & -6 \\ 3 & 0 & 18 \end{bmatrix}}_{3B} - \underbrace{\begin{bmatrix} -8 & 2 & 18 \\ 6 & -10 & 4 \end{bmatrix}}_{2A} = \underbrace{\begin{bmatrix} 29 & 7 & -24 \\ -3 & 10 & 14 \end{bmatrix}}_{3B-2A}$$

3.2 Matrix Multiplication

In certain cases, we can multiply two matrices. The process is more difficult than adding/subtracting matrices or multiplying by a scalar.

Because of its wide use in application problems, we need to learn multiplication of matrices well.

match

If A is a $n \times m$ matrix and B is an $m \times r$ matrix, then we can multiply to find the product matrix AB. The resulting matrix will be an $n \times r$ matrix.

$n \times r$

dimension of AB

Example 3: Let $A = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$.

Find AB

dimension $\underbrace{1 \times \underbrace{3 \quad \text{dimension} \quad 3}_{\text{match}} \times 1}_{\text{AB will have dimension } 1\times 1.}$

$$AB = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(-1) + 1(4) + (-3)(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \end{bmatrix}$$

3.2 Matrix Multiplication

Example 4: Let $A = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$.

Find BA

$$BA \to \underbrace{(3\times \underbrace{1)\cdot(1}_{\text{match} \ddot\smile}\times 3)}_{\text{output is a } 3\times 3 \text{ matrix}}$$

$$\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}\begin{bmatrix} 2 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -1(2) & (-1)(1) & (-1)(-3) \\ 4(2) & 4(1) & 4(-3) \\ 0 & 0 & 0 \end{bmatrix}$$

Example 5: Let $A = \begin{bmatrix} 2 & 1 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 4 & 1 \\ 3 & -2 \end{bmatrix}$.

Find AB

$$\underbrace{(1 \times \underbrace{3) \cdot (3}_{\text{match :)}} \times 2)}_{\text{output dimension is } 1\times 2}$$

$$\begin{bmatrix} 2 & 1 & -3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2(-1)+(1)(4)+(-3)(3) & 2(2)+(1)(1)+(-3)(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 11 \end{bmatrix}$$

3.2 Matrix Multiplication

Example 6: Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 3 & -2 \end{bmatrix}$.

Find AB

$$(2 \times \underbrace{3) \cdot (3}_{\text{match}} \times 2)$$

output is 2×2

$$AB = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & 4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} -8 & 16 \\ 12 & 18 \end{bmatrix}$$

3.2 Matrix Multiplication and Systems of Equations

In chapter 1, we discussed solving systems of equations. In this chapter, we will use matrices to solve systems of equations.

We can pass back and forth between the system of equation format we are used to seeing and **matrix equations**.

Example 7: Express the following system of equations as a matrix equation $AX = B$.

$$\begin{cases} 3x - 4y + 5z = 7 \\ 2x - 8y + 9z = -2 \\ 5x \qquad\quad - 6z = 7 \end{cases}$$

$$\begin{bmatrix} 3 & -4 & 5 \\ 2 & -8 & 9 \\ 5 & 0 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 7 \end{bmatrix}$$

3.3 Solving Systems Using the Gauss-Jordan Method

In this section we will learn about the Gauss-Jordan method for solving systems of equations.

The Gauss-Jordan method is applied to the **augmented matrix** that represents a system of equations.

First we need to learn how to pass back and forth between the system of equation format we are used to seeing and the corresponding augmented matrix.

3.3 Solving Systems Using the Gauss-Jordan Method

Consider the system of equations

$$\begin{cases} 2x - 4y + 7z = 5 \\ 8x - 2y + 2z = -9 \\ 5x + 6y - 6z = 7 \end{cases}$$

The corresponding augmented matrix is

$$\begin{array}{c} \begin{array}{cccc} x & y & z = & \text{constants} \end{array} \\ \left[\begin{array}{ccc|c} 2 & -4 & 7 & 5 \\ 8 & -2 & 2 & -9 \\ 5 & 6 & -6 & 7 \end{array}\right] \end{array}$$

Each row of the matrix represents one of the equations. The first three columns are the coefficients of x, y, and z respectively. The vertical line represents the equal sign and the 4th column contains the constants.

The Gauss-Jordan method is a systematic use of the elimination method. In this next example, we will solve a system of equations using the elimination method while keeping track of the corresponding augmented matrix at each step.

Example 1: Solve the system

$$-3\left(\begin{cases} x + 3y = 7 \\ 3x + 4y = 11 \end{cases}\right) \qquad \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 3 & 4 & 11 \end{array}\right]$$

$$\begin{array}{r} -\cancel{3x} - 9y = -21 \\ \cancel{3x} + 4y = 11 \\ \hline -5y = -10 \end{array} \implies \begin{array}{r} x + 3y = 7 \\ -5y = -10 \end{array} \qquad \left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & -5 & -10 \end{array}\right]$$

3.3 Example 1 continued

$$\begin{aligned} x + 3y &= 7 \\ \frac{-5y}{-5} &= \frac{-10}{-5} \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x + 3y &= 7 \\ (y &= 2)\,{-3} \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 3 & 7 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x + \cancel{3y} &= 7 \\ -\cancel{3y} &= -6 \\ \hline x \quad &= 1 \end{aligned}$$

$$\begin{aligned} x &= 1 \\ y &= 2 \end{aligned}$$

$$\begin{array}{ccc} x & y & \text{constants} \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

Using the elimination method is repetitive use of three operations. When we apply the operations to the corresponding augmented matrix, we are using what are called **row operations**.

Row Operations

1. Any two rows in the augmented matrix can be interchanged.
2. Any row may be multiplied by a non-zero constant.
3. A constant multiple of a row may be added to another row.

Each of these operations make the system "look" different, but create an equivalent system of equations (the solution does not change).

Example 2: Solve the system of equations by applying row operations to the corresponding augmented matrix.

$$\begin{cases} 2x + y + 2z = 10 \\ x + 2y + z = 8 \\ 3x + y - z = 2 \end{cases}$$

- Write the augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 1 & 2 & 10 \\ 1 & 2 & 1 & 8 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

3.3 Example 2 continued

- Interchange rows if necessary to obtain a non-zero entry in the first row, first column.

Already done!

- Use a row operation to make the entry in the first row, first column a 1.

$R_1 \longleftrightarrow R_2$

swap rows

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & 2 & 10 \\ 3 & 1 & -1 & 2 \end{array}\right]$$

- Use row operations to make all other entries in the first column zero.

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -3 & 0 & -6 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

$-3y = -6$

- Interchange rows if necessary to obtain a nonzero number in the second row, second column.

Not needed

- Use row operations to make the entry in the second row, second column a 1.

$R_2 \to -\frac{1}{3}R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 2 \\ 0 & -5 & -4 & -22 \end{array}\right]$$

- Use row operations to make all other entries in the second column zero.

$R_1 \to R_1 - 2R_2$

$R_3 \to R_3 + 5R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

- Use row operations to make the entry in the third row, third column a 1 and all other entries in the third column zero. Read off the answer from the augmented matrix.

$R_3 \to -\frac{1}{4}R_3$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$R_1 \to R_1 - R_3$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$(x, y, z) = (1, 2, 3)$

3.3 The Gauss-Jordan Method

Example 2 outlined the Gauss-Jordan method. You can also find the steps on page 54 of the text.

The process of making a certain entry in a matrix 1 and then all other entries in that column zero is called **pivoting**.

The number that is made a 1 is called the **pivot element** and the row that contains the pivot element is called the **pivot row**.

The final form of the augmented matrix after applying the Gauss-Jordan method is called **reduced row echelon form**.

Example 3: Solve the system of equations using the Gauss-Jordan method.

$$\begin{cases} x - y - z = -1 \\ x - 3y + 2z = 7 \\ 2x - y + z = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 1 & -3 & 2 & 7 \\ 2 & -1 & 1 & 3 \end{array}\right] \xrightarrow[R_3 \longrightarrow R_3 - 2R_1]{R_2 \longrightarrow R_2 - 1 \cdot R_1} \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & -2 & 3 & 8 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

make these zero using row 1

3.3 Example 3 continued

$$R_2 \longleftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & -1 & -1 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & -2 & 3 & 8 \end{array}\right]$$

make zero by using row 2

$$\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 9 & 18 \end{array}\right]$$

3.3 Example 3 continued

$$R_3 \longrightarrow \frac{1}{9}R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 - 2R_3} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\Rrightarrow (x, y, z) = (0, -1, 2)$$

3.4 Systems of Equations - Special Cases

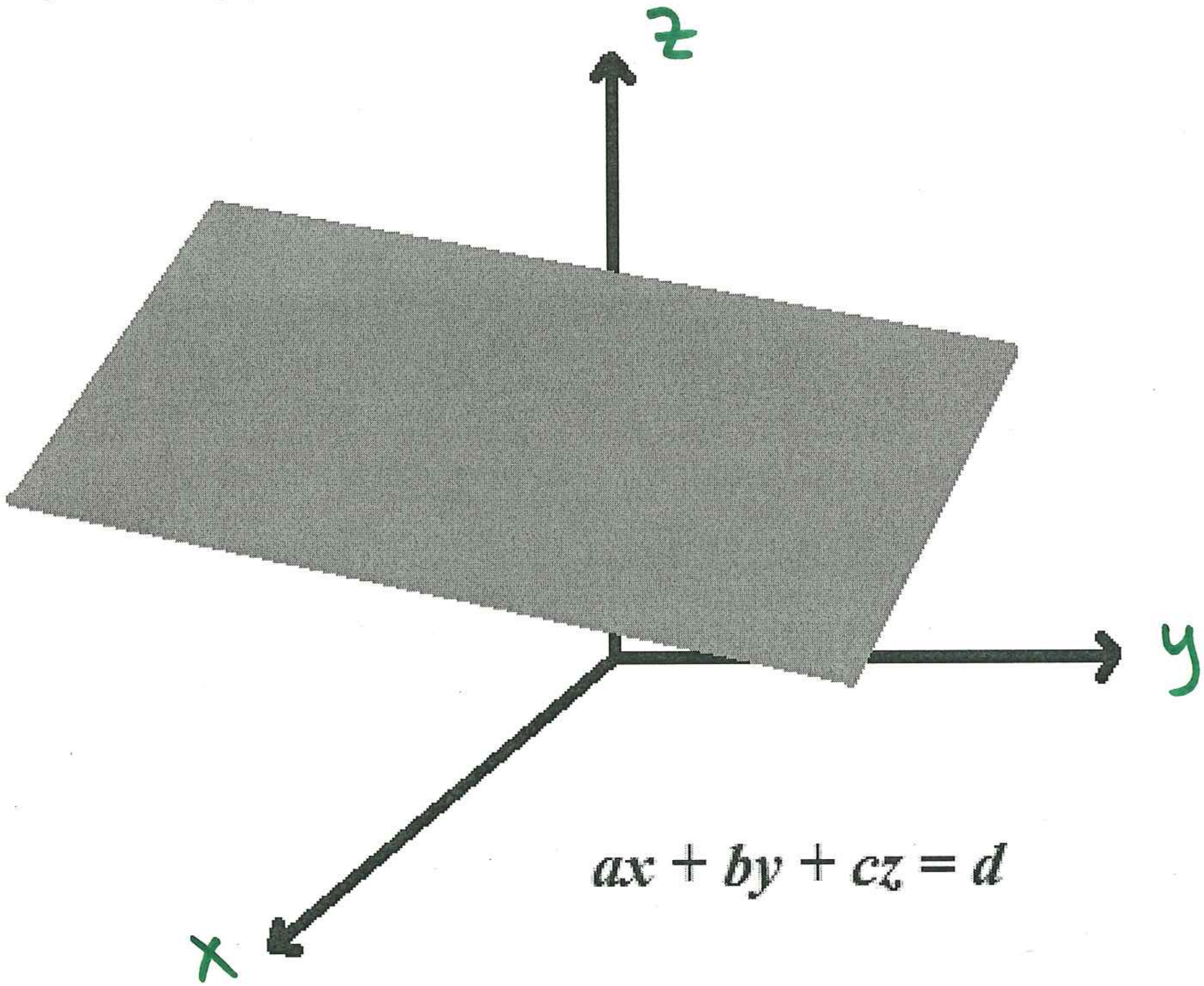
In Chapter 1, we considered the intersection of two lines in a plane and saw that three things could happen.

1. The lines intersect at exactly one point. This is called an **independent system.**
2. The lines are parallel with different y-intercepts so they do not intersect. This is called an **inconsistent system**.
3. The lines coincide and intersect at infinitely many points. This is called a **dependent system**.

If a system has at least one solution, then we called it a **consistent system**. A consistent system is one that is either independent or dependent.

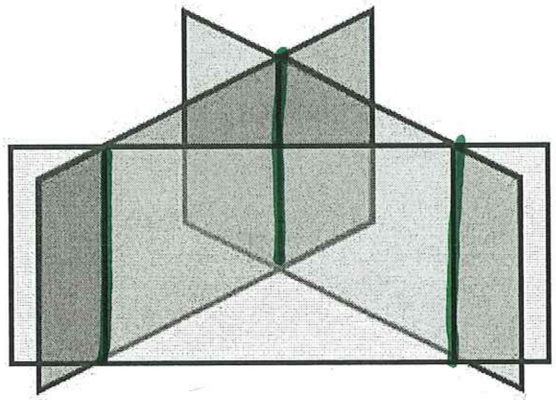
3.4 Systems of Equations - Special Cases

Now we are looking at systems with three or more variables. In the case of a linear equation with three variables, the set of all points that satisfy the equation is a plane in 3-dimensions.

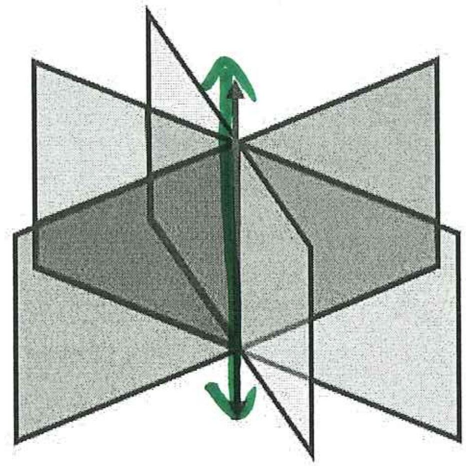


3.4 Systems of Equations - Special Cases

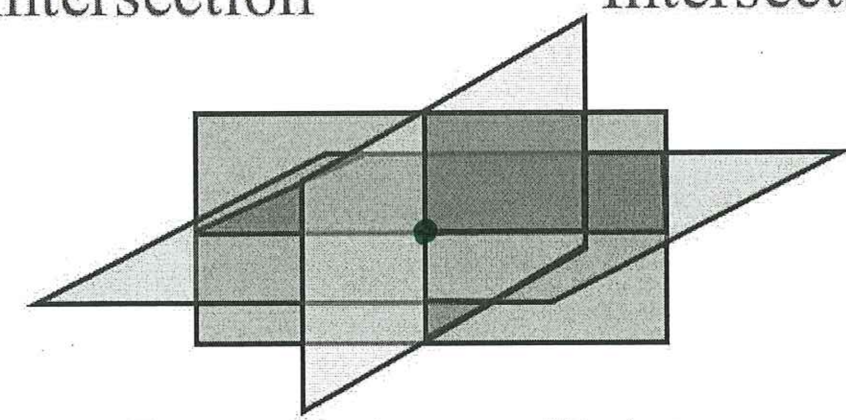
When we want to solve a system of 3 equations with 3 variables, we need to know how the planes intersect. There are a variety of ways that three planes can intersect. Here are a few examples.



No Intersection



Intersects at a Line



Intersects at a Point

3.4 Systems of Equations - Special Cases

We use the same words to describe solutions to systems of equations with three variables as we did with lines.

In the last slide, we saw

- No intersection - Inconsistent

  No Solution

- Intersects at a line - Dependent (consistent)

  Infinitely many

- Intersects at a point - Independent (consistent)

  Exactly one.

3.4 Systems of Equations - Special Cases

In Chapter 3, we are using the Gauss-Jordan method to solve systems of equations by applying the method to the augmented matrix until the matrix is in reduced row echelon form.

Now we need to see what happens when we apply the Gauss-Jordan method to augmented matrices corresponding to inconsistent and dependent systems.

Example 1: Determine all solutions (if any exist) for the system of equations

$$\begin{cases} 2x + 3y = 7 \\ -4x - 6y = -13 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ -4 & -6 & -13 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_1} \begin{array}{ccc} x & y & \text{constant} \\ \left[\begin{array}{cc} 2 & 3 \\ 0 & 0 \end{array}\right. & & \left|\begin{array}{c} 7 \\ 1 \end{array}\right] \end{array}$$

make zero using R_1

$0x + 0y = 1$

$0 = 1$

Never True

$\Longrightarrow$ No Solution

Example 2: Determine all solutions (if any exist) for the system of equations

$0x + 0y = 0$

$$\begin{cases} 2x + 3y = 7 \\ -4x - 6y = -14 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ -4 & -6 & -14 \end{array}\right] \xrightarrow[R_2 + 2R_1]{} \left[\begin{array}{cc|c} 2 & 3 & 7 \\ 0 & 0 & 0 \end{array}\right]$$

All x values are valid.

$2x + 3y = 7$

$3y = -2x + 7$

$y = -\frac{2}{3}x + \frac{7}{3}$

$(x, y) = \left(x, -\frac{2}{3}x + \frac{7}{3}\right)$

parameterized solution.

Example 3: Determine all solutions (if any exist) for the system of equations

$$\begin{cases} x + y + z = 6 \\ 3x + 2y + z = 14 \\ 4x + 3y + 2z = 20 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 3 & 2 & 1 & 14 \\ 4 & 3 & 2 & 20 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-3R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -1 & -2 & -4 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

make zero

3.4 Example 3 continued

$$R_2 \to -R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$x - z = 2 \Rightarrow x = z + 2$$

$$y + 2z = 4 \Rightarrow y = -2z + 4$$

$$(x, y, z) = (z + 2, -2z + 4, z)$$

Example 4: Determine all solutions (if any exist) for the system of equations

$$\begin{cases} x + 2y - 4z = 1 \\ 2x - 3y + 8z = 9 \end{cases}$$

Already a 1 :)

make this a 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 2 & -3 & 8 & 9 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & -7 & 16 & 7 \end{array}\right]$$

Make zero using R_1

$$\xrightarrow{R_2 \to -\frac{1}{7} R_2} \left[\begin{array}{ccc|c} 1 & 2 & -4 & 1 \\ 0 & 1 & -\frac{16}{7} & -1 \end{array}\right]$$

make zero using R_2 on next page

$$R_1 \to R_1 - 2R_2 \quad \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 4/7 & 3 \\ 0 & 1 & -\frac{16}{7} & -1 \end{array}\right] \end{array}$$

$$-4 - 2\left(-\frac{16}{7}\right) = -4 + \frac{32}{7} = -\frac{28}{7} + \frac{32}{7} = \frac{4}{7}$$

Done reducing.

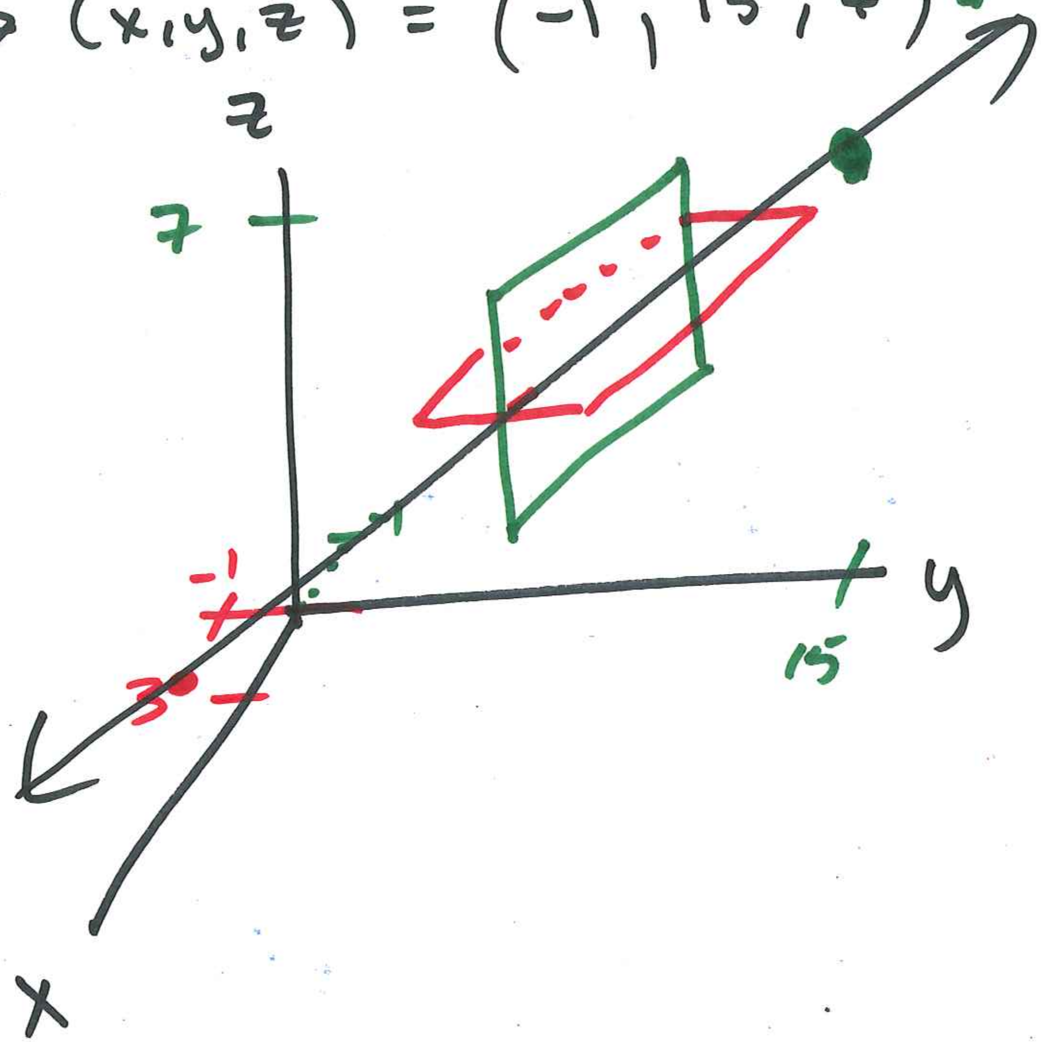
$$x + \frac{4}{7}z = 3 \implies x = -\frac{4}{7}z + 3$$

$$y - \frac{16}{7}z = -1 \implies y = \frac{16}{7}z - 1$$

$$\implies (x, y, z) = \left(-\frac{4}{7}z + 3,\ \frac{16}{7}z - 1,\ z\right)$$

$z = 0 \implies (x, y, z) = (3, -1, 0)$

$z = 7 \implies (x, y, z) = (-1, 15, 7)$



3.4 Systems of Equations - Special Cases

1. If any row of the reduced row echelon form of the matrix gives a false statement such as $0 = 1$, then the system has no solution.

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 5 \end{array}\right]$$

2. If the reduced row echelon form has fewer equations than variables and the system is consistent, then there are infinitely many solutions (rows containing all zeros dropped).

   - If a system has an infinite number of solutions, the solution must be put in **parameterized** form.
   - The number of arbitrary parameters equals the number of variables minus the number of non-zero equations.

Ex 4: 3 variables − 2 equations = 1 parameter
we used z

$2 \cdot \frac{1}{2} = 1 \qquad 2 \cdot 2^{-1} = 1$

Definition: A $n \times n$ matrix A has an inverse if there exists another $n \times n$ matrix B such that $AB = BA = I_n$ where I_n is the $n \times n$ identity matrix.

The inverse of a matrix A, if it exists, is denoted by A^{-1}.

Note: If is important that you check both $AB = I_n$ and $BA = I_n$. Just checking one is not enough since in general $AB \neq BA$.

Example 1: Verify that $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ are inverses.

check ① $AB = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 3(2)-5(1) & 3(-5)+5(3) \\ 1(2)-2(1) & 1(-5)+2(3) \end{bmatrix} \doteq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

② $BA = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

If you look closely at the last example, you will see a relationship between the two matrices.

If the determinant is zero, A is not invertible.

For a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the inverse matrix is

determinant

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Unfortunately, there is no shortcut for larger matrices.

Method for Finding the Inverse of a Matrix

1. Write the augmented matrix $\left[A \,\middle|\, I_n\right]$.

2. Use the Gauss-Jordan method to write the augmented matrix from step 1 in reduced row echelon form.

3. If the reduced row echelon form in step 2 is $\left[I_n \,\middle|\, \underset{A^{-1}}{\underset{\shortparallel}{B}}\right]$, then B is the inverse of A.

4. If the left side of the reduced matrix from step 2 is not an identity matrix, then A does not have an inverse.

Example 2: Find the inverse of the matrix

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 7 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1(1/7)$

$R_2 \to R_2(-1/2)$

$R_3 \to R_3(1/4)$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/7 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/4 \end{array}\right]$$

$= A^{-1}$

Example 3: Assume $x, y, z \neq 0$ and find the inverse of the matrix

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} x & 0 & 0 & 1 & 0 & 0 \\ 0 & y & 0 & 0 & 1 & 0 \\ 0 & 0 & z & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R_2 \to R_2(1/y) \\ R_3 \to R_3(1/z)}]{R_1 \to R_1(1/x)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{x} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/y & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/z \end{array}\right]$$

Example 4: Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

Already = 1

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -3 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

make this a 1

Make zero using R_1

$$\xrightarrow{R_2 \to -\frac{1}{2} R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 3/2 & -1/2 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

make zero using R_2

$$R_1 \to R_1 - R_2 \quad \longrightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 3/2 & -1/2 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

make a 1

$$R_3 \to \tfrac{1}{3} R_3 \quad \longrightarrow \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/2 & 1/2 & 0 \\ 0 & 1 & -1 & 3/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 1/3 \end{array}\right]$$

make zero using R_3

3.5 Example 4 continued

$$\xrightarrow{R_2 \to R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{7}{6} & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{7}{6} & -\frac{1}{2} & \frac{1}{3} \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix}$$

3.5 Using the Inverse to Solve a System of Equations

Suppose we want to solve the system $AX = B$ where A is the coefficient matrix, X is the matrix of variables and B is the matrix of constant terms.

If A is invertible, then we can multiply both sides of the equation $AX = B$ by A^{-1} to get

$$\underbrace{A^{-1}A}_{I}X = A^{-1}B \implies X = A^{-1}B$$

Note: This method only works when a unique solution exists.

3.5 Using the Inverse to Solve a System of Equations

Example 5: Solve the system of equations using the inverse method.

$$A \cdot X = B$$

$$\begin{cases} 3x - 5y = 2 \\ -x + 2y = 0 \end{cases} \Rightarrow \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6-5} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \qquad (x, y) = (4, 2)$$

Example 1: Let $X = \begin{bmatrix} 2x & 5y \\ 3 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 4 \\ z & 3w \end{bmatrix}$.

Find $2X - 3Y$.

$$\underbrace{\begin{bmatrix} 4x & 10y \\ 6 & 10 \end{bmatrix}}_{2X} - \underbrace{\begin{bmatrix} -3 & 12 \\ 3z & 9w \end{bmatrix}}_{3Y}$$

$$= \begin{bmatrix} 4x+3 & 10y-12 \\ 6-3z & 10-9w \end{bmatrix}$$

Example 2: Let $X = \begin{bmatrix} 2x & 5y \\ 3 & 5 \end{bmatrix}$ and $Y = \begin{bmatrix} -1 & 4 \\ z & 3w \end{bmatrix}$.

Find XY.

$$\begin{bmatrix} 2x & 5y \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ z & 3w \end{bmatrix}$$

$$= \begin{bmatrix} -2x + 5zy & 8x + 15yw \\ -3 + 5z & 12 + 15w \end{bmatrix}$$

3.7 Applications - Leontief Models

In the 1930's, Wassily Leontief used matrices to model economic systems.

His models, referred to as input-output models, divide the economy into sectors where each sector produces goods and services for itself and other sectors.

The sectors are dependent on one another and the total input always equal the total output.

In 1973, he won the Nobel Prize in Economics for his work.

3.7 The Closed Model

As an example of the closed model, we look at a very simple economy were there are only three sectors.

Example 1: Assume that in a village there is a farmer, carpenter, and tailor who provide the three essential goods: food, shelter, and clothing. Suppose the farmer consumes 30% of the food he produces, and gives 25% to the carpenter, and 45% to the tailor. 40% of the carpenter's production is consumed by the farmer, 35% by himself, and 25% by the tailor. 35% of the tailor's production is consumed by the farmer, 15% by the carpenter, and 50% by himself. Write the matrix that describes the closed model.

3.7 Example 1 continued

	Farmer Production	Carpenter	Tailor
Consumption by Farmer	.30	.40	.35
Carpenter	.25	.35	.15
tailor	.45	.25	.50

$$A = \begin{bmatrix} .30 & .40 & .35 \\ .25 & .35 & .15 \\ .45 & .25 & .50 \end{bmatrix}$$

Internal Consumption Matrix

-or-

Exchange Matrix

Can also let x = farmers production, y = carpenters
z = tailors

Example 2: Using the information from example 1, how much pay should each person get for their effort?

X = farmers pay , y = Carpenters pay , z = tailors pay

Amount consumed = Amount producted

$$.30x + .40y + .35z = x$$
$$.25x + .35y + .15z = y$$
$$.45x + .25y + .50z = z$$

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} - A\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Ex 2 cont.

$$I \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} - A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow (I - A) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .30 & .40 & .35 \\ .25 & .35 & .15 \\ .45 & .25 & .5 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} .7 & -.4 & -.35 & 0 \\ -.25 & .65 & -.15 & 0 \\ -.45 & -.25 & .5 & 0 \end{array} \right] \sim$$

row operation can be found on website linked in Canvas announcement

$$
\begin{array}{ccc|c}
x & y & z & \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & -\frac{115}{142} & 0 \\
0 & 1 & -\frac{77}{142} & 0 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

$$x - \frac{115}{142} z = 0$$

$$y - \frac{77}{142} z = 0$$

$$x = \frac{115}{142} z$$

$$y = \frac{77}{142} z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{115}{142} z \\ \frac{77}{142} z \\ z \end{bmatrix}$$

This will always be given to you

If $z = 142$

$\Rightarrow$ $x = 115$

$y = 77$

I picked $z = 142$ because it made the numbers work out nicely

3.7 The Open Model

The open model is more realistic, as it deals with an economy where sectors of the economy not only satisfy each others needs, but also satisfy some external demand.

The external demand comes from the consumer. The basic assumption is the same, whatever is produced is consumed.

Example 3: Consider the same economy as in example 1, however the production of the farmer, carpenter and tailor is not completely consumed by the three sectors. Instead, the consumption is given by the following table.

consumption

production

	F produces	C produces	T produces
F uses	.15	.20	.15
C uses	.20	.25	.20
T uses	.10	.15	.05

Internal consumption matrix -or- exchange

Also, assume that the consumer uses the other \$50000 worth of food, \$40000 worth of carpenter's production, and \$35000 worth of clothing.

What should be the amount, in thousands of dollars, of required output by each sector to meet the external demand?

Exchange Matrix $A = \begin{bmatrix} .15 & .20 & .15 \\ .20 & .25 & .20 \\ .10 & .15 & .05 \end{bmatrix}$

Want production = consumption

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = Ax + \begin{bmatrix} 50 \\ 40 \\ 35 \end{bmatrix}$$

$D =$ external demand matrix

Solve $\boxed{X = Ax + D}$

x = output of farmer
y = output of carpenter
z = output of Tailor

Example 3: Solve the system of equations using Gauss-Jordan elimination.

$$\begin{cases} x + 2y + 3z = 9 \\ 2x - y + z = 8 \\ 3x \qquad - z = 3 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

Example 3 continued

$$\xrightarrow{R_2 \to -\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 1 & 2 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 6R_2]{R_1 \to R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

Example 3 continued

$$R_3 \to -\frac{1}{4}R_3 \quad \longrightarrow \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 - R_3]{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$(x, y, z) = (2, -1, 3)$$

Example 4: Determine the solution(s) to the system of equations using Gauss-Jordan elimination.

$$\begin{cases} x - y + z = 0 \\ 4x + 2y - 6z = 0 \\ 2x + y - 3z = 0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 4 & 2 & -6 & 0 \\ 2 & 1 & -3 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 6 & -10 & 0 \\ 0 & 3 & -5 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 \to 2R_3 - R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 6 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Example 4 continued

$$\xrightarrow{R_2 \to \frac{1}{6}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x - 2/3\, z = 0 \\ y - 5/3\, z = 0 \end{array}$$

$$\Longrightarrow \quad \begin{array}{l} x = \frac{2}{3} z \\ y = \frac{5}{3} z \end{array} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/3\, z \\ 5/3\, z \\ z \end{bmatrix} = z \begin{bmatrix} 2/3 \\ 5/3 \\ 1 \end{bmatrix}$$

Example 5: Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 1 & x \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

cannot be zero

$$A = \begin{bmatrix} 3 & 4 \\ 1 & x \end{bmatrix} \qquad A^{-1} = \frac{1}{3x-4} \begin{bmatrix} x & -4 \\ -1 & 3 \end{bmatrix}$$

$3x - 4 \neq 0$

Extra Examples

Example 6: Tommy has to buy some pens, pencils and notebooks for the upcoming semester. He has \$102 to spend on \$5 pens, \$3 pencils, and \$9 notebooks. He would like to spend the same amount of money on pens as on pencils. He also wants the combined number of pens and pencils to be equal to that of notebooks. Set up a system of equations to find how many of each item he should buy.

x = # of pencils bought

y = # of pens bought

z = # of notebook bought

Example 6 continued

$3x + 5y + 9z = 102$ ← total \$ spent

$3x = 5y$ $(3x - 5y = 0)$ ← \$ spent on pens = \$ spent on pencils

$x + y = z$ $(x + y - z = 0)$

$$\left[\begin{array}{ccc|c} 3 & 5 & 9 & 102 \\ 3 & -5 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{bmatrix} 3 & 5 & 9 \\ 3 & -5 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 102 \\ 0 \\ 0 \end{bmatrix}$$

Example 7: John inherited \$25,000 and invested part of it in a money market account, part in municipal bonds, and part in a mutual fund. After one year, he received a total of \$1,620 in simple interest from the three investments. The money market paid 6% annually, the bonds paid 7% annually, and the mutually fund paid 8% annually. There was \$6,000 more invested in the bonds than the mutual funds. Set up a system of equations to find the amount John invested in each category.

X = amount of \$ invested in money market
Y = " " municipal bonds
Z = " " mutual funds

Example 7 continued

$$.06x + .07y + .08z = 1620$$ ← interest gained

$$x + y + z = 25000$$ ← total money invested

$$y = z + 6000 \quad (y - z = 6000)$$ ← ~~purple~~ **black** underlined

$$\left[\begin{array}{ccc|c} .06 & .07 & .08 & 1620 \\ 1 & 1 & 1 & 25000 \\ 0 & 1 & -1 & 6000 \end{array}\right] \qquad \begin{bmatrix} .06 & .07 & .08 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1620 \\ 25000 \\ 6000 \end{bmatrix}$$

3.7 The Open Model

Example 3: Consider the same economy as in example 1, however the production of the farmer, carpenter and tailor is not completely consumed by the three sectors. Instead, the consumption is given by the following table.

consumption

production

	F produces	C produces	T produces
F uses	.15	.20	.15
C uses	.20	.25	.20
T uses	.10	.15	.05

Internal consumption matrix
-or-
exchange

Also, assume that the consumer uses the other \$50000 worth of food, \$40000 worth of carpenter's production, and \$35000 worth of clothing.

What should be the amount, in thousands of dollars, of required output by each sector to meet the external demand?

3.7 Example 3 continued

Exchange Matrix $A = \begin{bmatrix} .15 & .20 & .15 \\ .20 & .25 & .20 \\ .10 & .15 & .05 \end{bmatrix}$

Want production = consumption

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = X = AX + \begin{bmatrix} 50 \\ 40 \\ 35 \end{bmatrix}$$

D = external demand matrix

Solve $\boxed{X = AX + D}$

x = outpot of farmer
y = output of carpenter
z = output of Tailor

Solve $X = AX + D$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I \cdot X - AX = D$$

$$(I-A)^{-1}(I-A)X = (I-A)^{-1}D$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \boxed{X = (I-A)^{-1}D}$$

$$I - A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .15 & .20 & .15 \\ .20 & .25 & .20 \\ .10 & .15 & .05 \end{bmatrix} = \begin{bmatrix} .85 & -.2 & -.15 \\ -.2 & .75 & -.2 \\ -.1 & -.15 & .95 \end{bmatrix}$$

$$(I-A)^{-1} = \begin{bmatrix} \frac{260}{199} & \frac{1700}{4179} & \frac{1220}{4179} \\ \frac{80}{199} & \frac{6340}{4179} & \frac{1600}{4179} \\ \frac{40}{199} & \frac{1180}{4179} & \frac{4780}{4179} \end{bmatrix}$$

calculator

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (I-A)^{-1} D$$

$$= (I-A)^{-1} \begin{bmatrix} 50 \\ 40 \\ 35 \end{bmatrix} = \begin{bmatrix} \frac{127900}{1393} \\ \frac{131200}{1393} \\ \frac{85500}{1393} \end{bmatrix}$$

$$= \begin{bmatrix} 91.816 \\ 94.185 \\ 61.378 \end{bmatrix} \rightarrow$$

Farmers \$91,816

Carpenters \$ 94,185

Tailors \$ 61,378

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

① $I - A = \begin{bmatrix} .6 & -.2 \\ -.5 & .5 \end{bmatrix}$

② $(I-A)^{-1} = \frac{1}{.3-.1} \begin{bmatrix} .5 & .2 \\ .5 & .6 \end{bmatrix} = 5 \begin{bmatrix} .5 & .2 \\ .5 & .6 \end{bmatrix}$

$= \begin{bmatrix} 2.5 & 1 \\ 2.5 & 3 \end{bmatrix}$