MA 162 : Finite Mathematics - Chapter 5

Linear Programming - A Geometric Approach

University of Kentucky

5.2 Maximization Applications

A typical **linear programming** problem consists of finding a maximum or minimum of a linear function subject to certain conditions.

The problems are classified as **maximization** or **minimization problems** or just **optimization problems**.

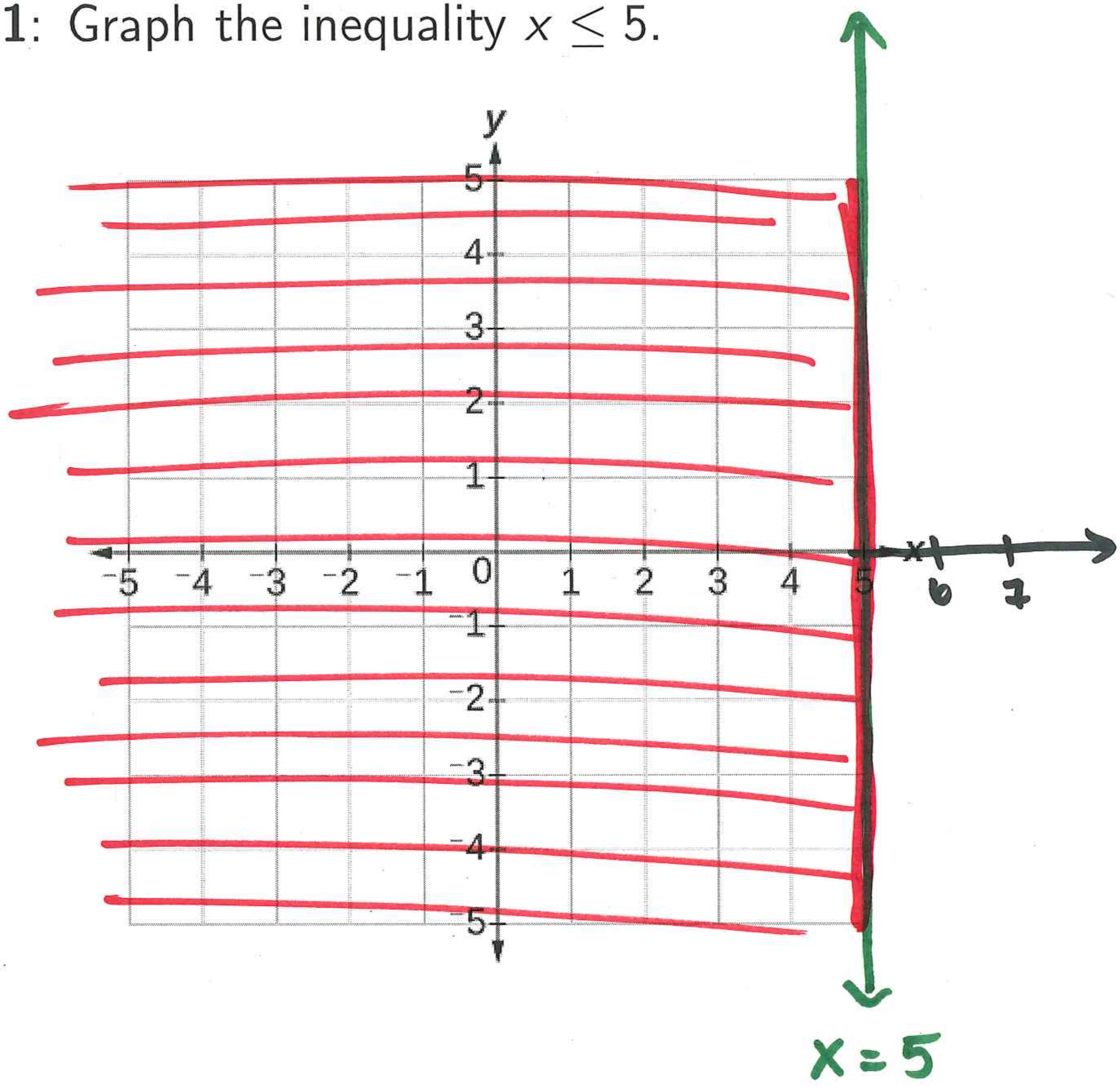
The function we are trying to optimize is called an **objective** function.

The conditions that must be satisfied are called **contraints**. The constraints usually come in the form of linear inequalities.

Before we solve any optimization problems, we need to practice graphing linear inequalities.

5.2 Maximization Applications

Example 1: Graph the inequality $x \leq 5$.



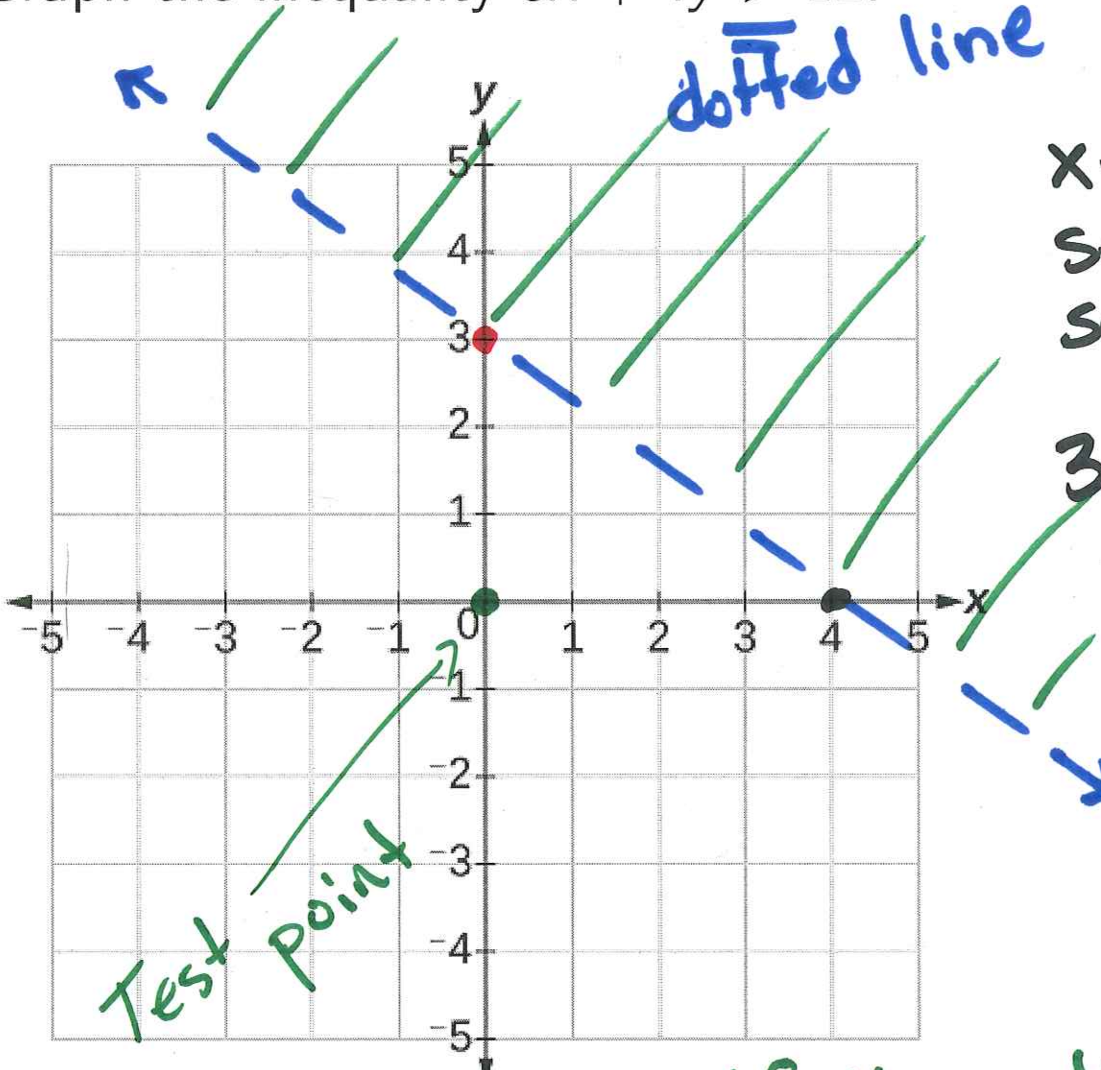
5.2 Maximization Applications

Example 2: Graph the inequality $3x + 4y > 12$.

Graph
$3x+4y=12$

y-int:
set $x=0$
solve for y

$3(0)+4y=12$
$4y=12$
$y=3$
$(0,3)$



x-int:
set $y=0$
solve for x

$3x+4(0)=12$
$3x=12$
$x=4$
$(4,0)$

Use $(0,0)$ and see if it satisfies the inequality. $3(0)+4(0)>12$
$0>12 \rightarrow$ False

5.2 Maximization Applications

Example 3: Graph the system of inequalities

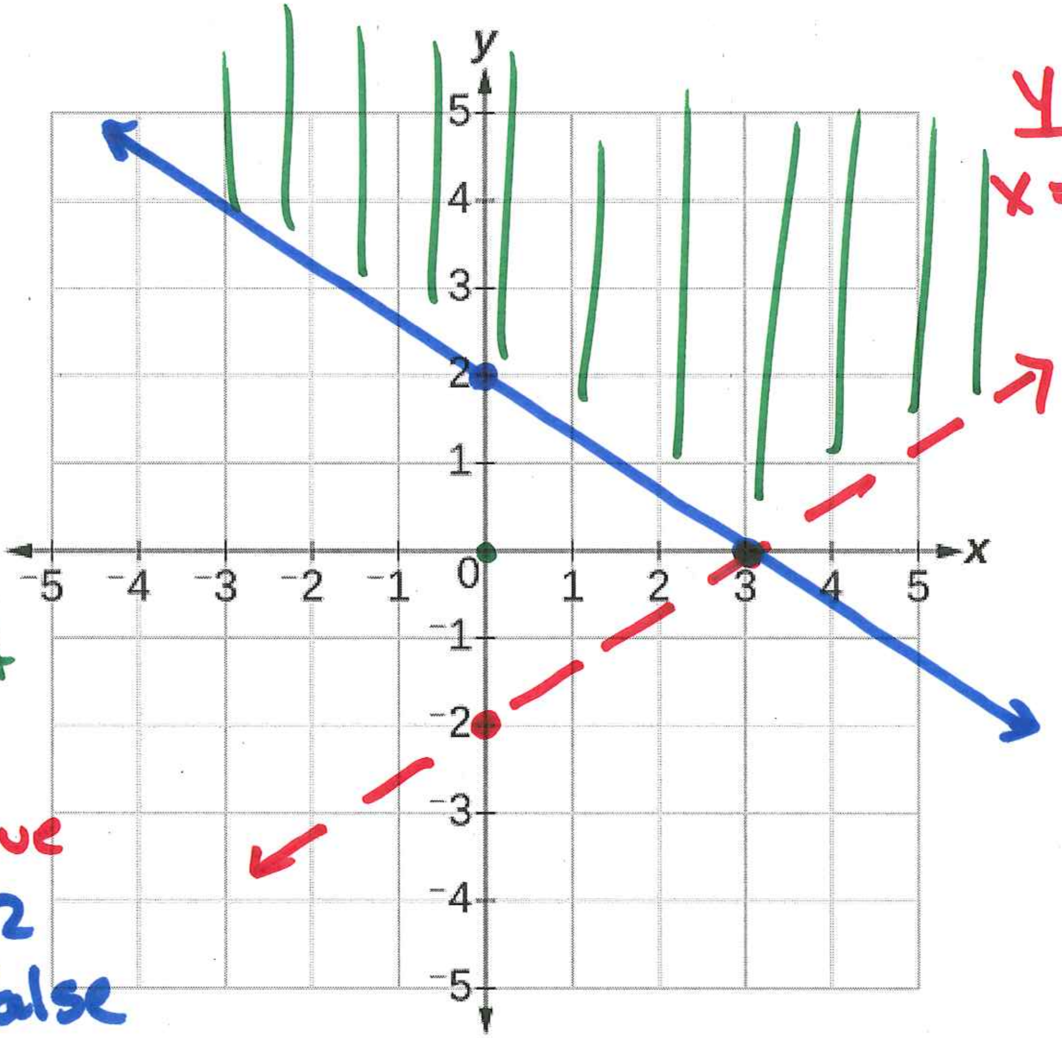
$$\begin{cases} 2x - 3y < 6 \\ 4x + 6y \geq 12 \end{cases}$$

x int:
$y=0 \Rightarrow 4x=12$
$x=3$
$(3,0)$

y int:
$x=0 \Rightarrow 6y=12$
$y=2$
$(0,2)$

x int:
$y=0 \Rightarrow 2x=6$
$x=3$
$(3,0)$

y int:
$x=0 \Rightarrow -3y=6$
$y=-2$
$(0,-2)$



(0,0) test point
$2(0)-3(0) < 6$
$0<6$ True
$4(0)+6(0) \geq 12$
$0 \geq 12$ False

5.2 Maximization Applications

The shaded region that satisfies all inequalities in a system of inequalities is called the **feasible region**.

The points that we use to determine where the feasible region is located are called **test points**.

The points of intersection of the graphed lines that are on the boundary of the feasible region are called **corner points**

Remark: Test points must not be located on any of the lines from the corresponding system of equations.

5.2 Maximization Applications

Example 4: Find the corner points of the feasible region for the system of inequalities

x int = (10,0)
y int = (0, 10)

$$\begin{cases} x + y \underline{\le} 10 \\ x - 3y \ge 6 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

x int = (6,0)
y int = (0,-2)

non-negativity constraints

↑
In 1st quadrant

$x + y \le 10$

$y \le -x + 10$

↑
below blue line

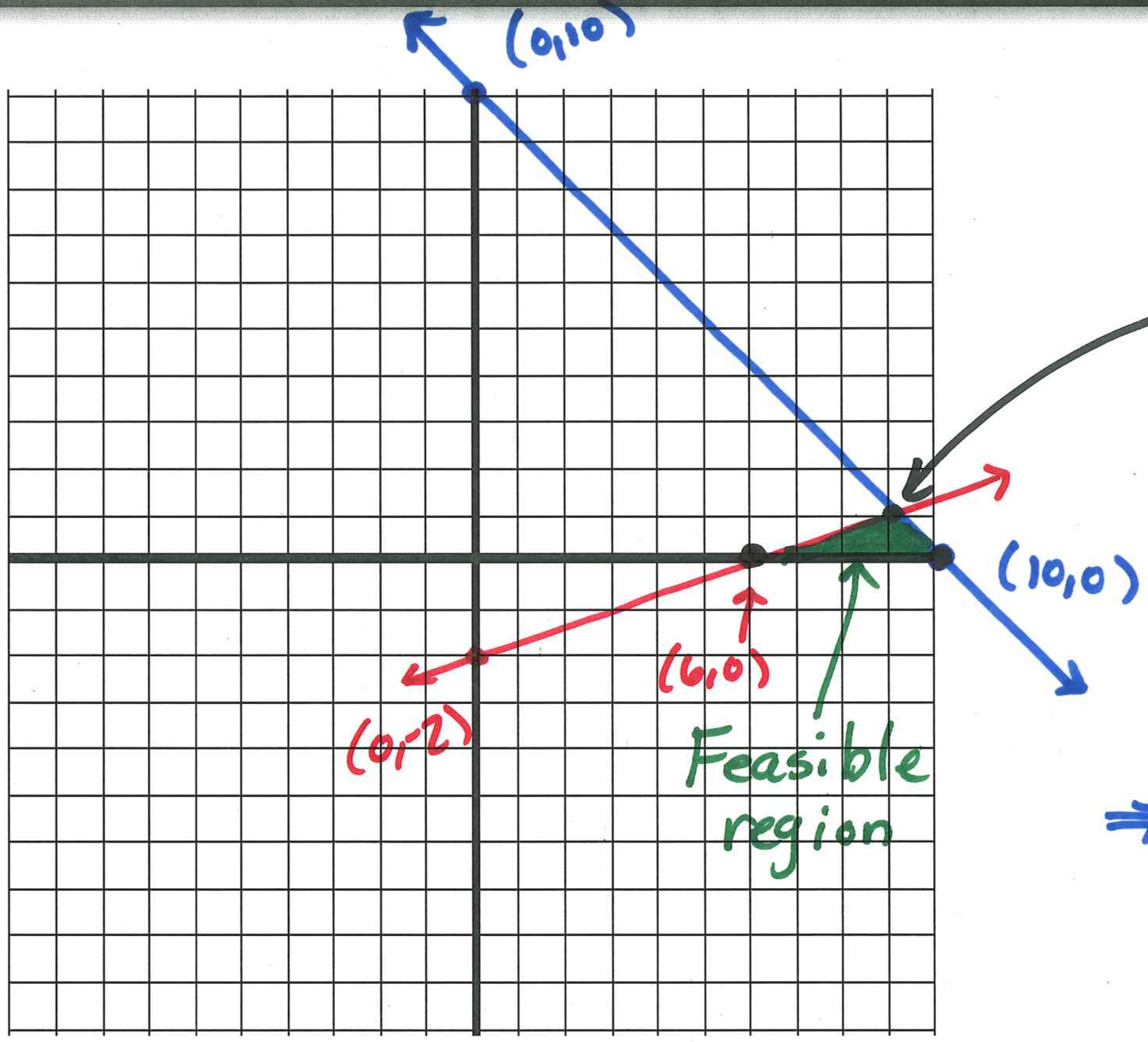
$x - 3y \ge 6$

$-3y \ge -x + 6$

$y \le \frac{x}{3} - 2$

↑
below red line

Example 4 continued



Find this point of intersection

Solve $\begin{cases} x+y=10 \\ x-3y=6 \end{cases}$

Top - bottom

$\Rightarrow 4y = 4$

$\boxed{y=1}$

$x + 1 = 10$

$\boxed{x=9}$

→ corner points are (6,0), (10,0), (9,1)

5.2 Maximization Applications

A linear programming problem consists of:

- An objective function (what are we trying to maximize/minimize?)
- Constraints (linear equalities or inequalities)

The goal of a linear programming problem is to maximize or minimize the objective function while satisfying all the constraints.

5.2 ~~Maximization~~ Applications

Minimization

Example 5: Set up the linear programming problem but do not solve.

A farmer uses two types of fertilizers. A 50-lb bag of Fertilizer A contains 8 lb of nitrogen, 2 lb of phosphorus, and 4 lb of potassium. A 50-lb bag of Fertilizer B contains 5 lbs of each of nitrogen, phosphorus, and potassium. The minimum requirements for a field are 440 lb of nitrogen, 260 lb or phosphorus, and 360 lb of potassium. If a 50-lb bag of Fertilizer A costs $30 and a 50-lb bag of Fertilizer B costs $20, find the amount of each type of fertilizer the farmer should use to minimize his cost while still meeting the minimum requirements.

Example 5 continued

Objective: Minimize Cost

x = # of bags of A

y = # of bags of B

$$C = 30x + 20y$$

Subject to: List of constraints

$$8x + 5y \geq 440$$

$$2x + 5y \geq 260$$

$$4x + 5y \geq 360$$

$$x \geq 0, \quad y \geq 0$$

5.2 Maximization Applications

Example 6: Set up the linear programming problem but do not solve.

A financier plans to invest up to $2 million in three projects. she estimates that Project A will yield a return of 10%, Project B 15%, and Project C 20% on her investment. Because of the risks associated with the investments, she decided not to put more than 20% of her total investment in Project C. She also decided that her investments in Projects B and C should not exceed 60% of her total investment. Finally, she decided that her investment in Project A should be at least 60% of her investments in Projects B and C. How much should she invest in each project if she wishes to maximize the total returns on her investments?

together

Example 6 continued

Objective: Maximize total return

Let x = amount invested in project A

y = amount invested in project B

z = amount invested in project C

Maximize $R = .10x + .15y + .20z$

Constraints:

$$.20(x+y+z) \le z$$

$$[illegible](y+z) \le .60(x+y+z)$$

$$.60(y+z) \le x$$

$$x, y, z \ge 0$$

5.2/5.3 Maximization and Minimization Applications

Recall: A linear programming problem consists of:

- An objective function which we want to maximize or minimize.
- Constraints in the form of linear equalities or inequalities.

Today we will discuss how to find the optimal solution given the set of constraints in a graphical manner. In chapter 7, we will discuss how to do this algebraically.

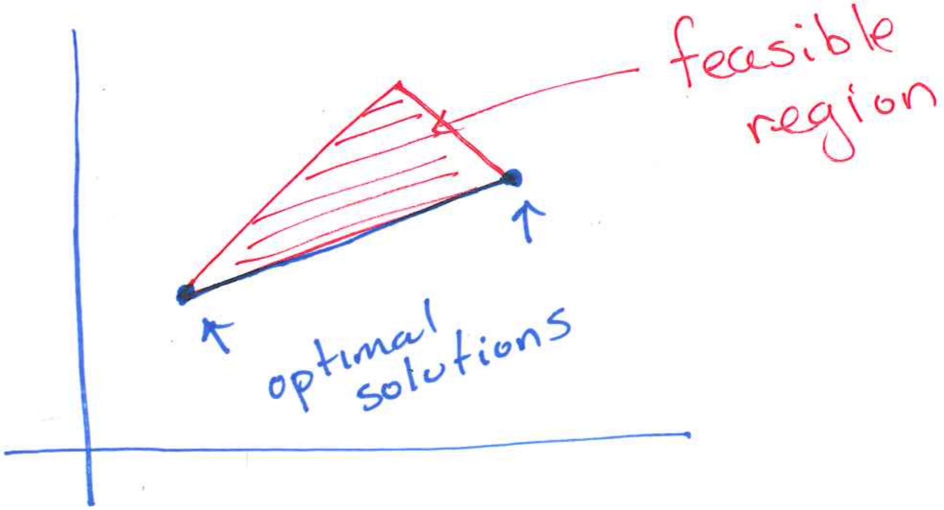
Terminology

- The set of all solutions to the system of constraints is called the **feasible region**.
- Any point in the feasible region is called a **feasible solution**.
- Any point outside the feasible region is **infeasible**.
- A point (if one exists) which optimizes the objective function is called an **optimal solution**.
- An optimal point must be feasible, but not every feasible point is optimal

Theorem - Solutions of Linear Programming Problems

If a linear programming problem has a solution, then it must occur at a corner point of the feasible region associated with the problem.

Furthermore, if the objective function is optimized at two adjacent corner points of the feasible region, then it is optimized at every point on the line segment connecting the two corner points. In this case, there are infinitely many solutions to the problem.



The Method of Corners

1. Graph the feasible set
2. Find the coordinates of all the corner points of the feasible set.
3. Evaluate the objective function at each corner point.
4. The optimal solution is the point which produces the largest (or smallest) value found in step 3.

5.2/5.3 Maximization and Minimization Applications

Example 1:A farmer uses two types of fertilizers. A 50-lb bag of Fertilizer A contains 8 lb of nitrogen, 2 lb of phosphorus, and 4 lb of potassium. A 50-lb bag of Fertilizer B contains 5 lbs of each of nitrogen, phosphorus, and potassium. The minimum requirements for a field are 440 lb of nitrogen, 260 lb or phosphorus, and 360 lb of potassium. If a 50-lb bag of Fertilizer A costs \$30 and a 50-lb bag of Fertilizer B costs \$20, find the amount of each type of fertilizer the farmer should use to minimize his cost while still meeting the minimum requirements.

x = # of bags of fertilizer A

y = # of bags of fertilizer B

Objective: Minimize $C = 30x + 20y$

Example 1 continued

Constraints:

$x \geq 0, y \geq 0$

	x-int	y-int
Nitrogen: $8x+5y \geq 440$	$(55,0)$	$(0,88)$
Phosphorus: $2x+5y \geq 260$	$(130,0)$	$(0,52)$
Potasium: $4x+5y \geq 360$	$(90,0)$	$(0,72)$

(0,0) does not satisfy any of these inequalities ⇒ shade on the opposite side

---

Nitrogen intersect Potasium

$$\begin{array}{r} 8x+5y=440 \\ -\ 4x+5y=360 \\ \hline 4x=80 \end{array}$$

$$x=20$$

plug into either equation

$$y=56 \quad \underline{\underline{(20,56)}}$$

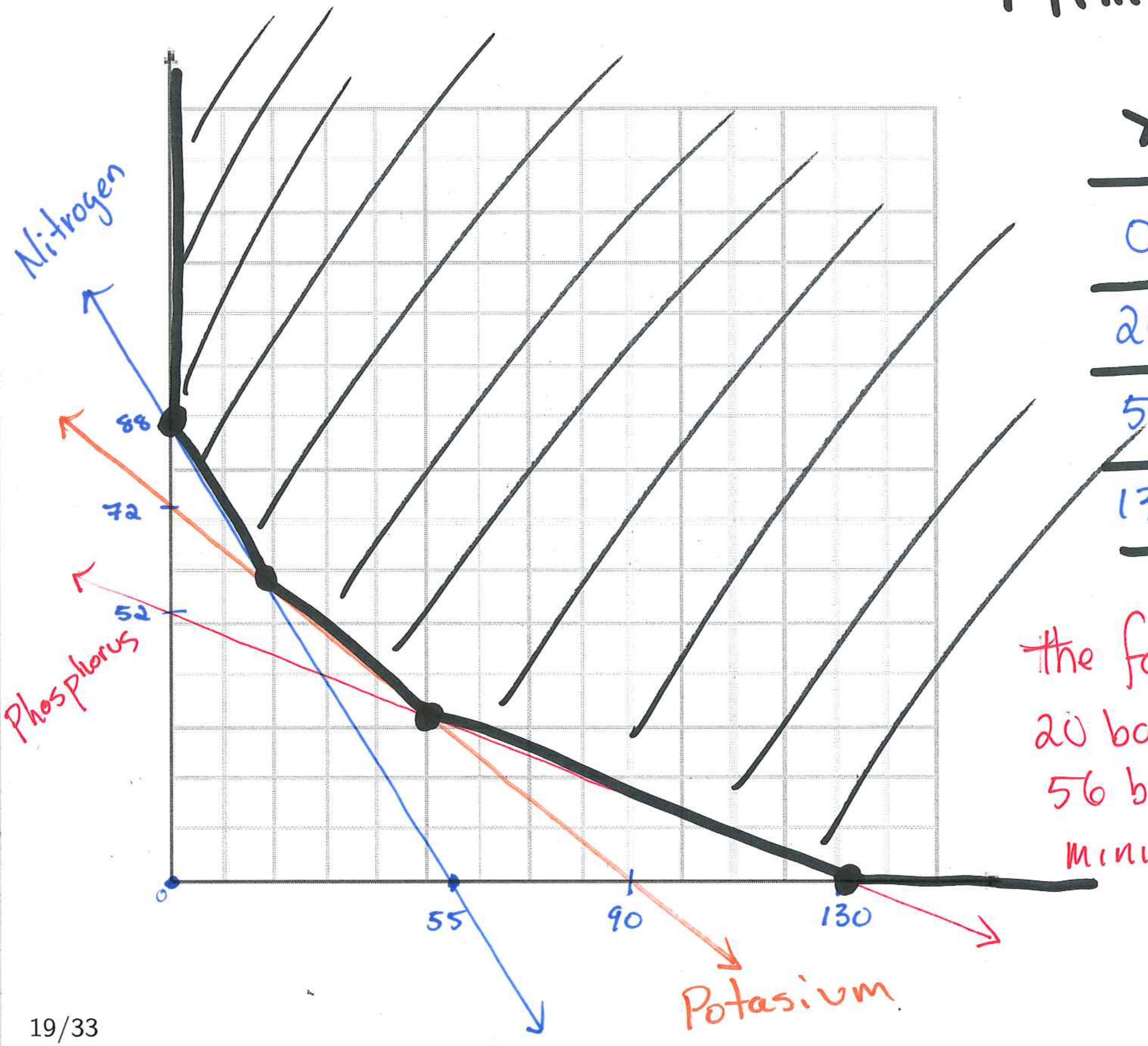
Phosphorus intersect Potasium

$$2x+5y=260$$

$$4x+5y=360$$

Reef #1 ⇒ $(50,32)$

Example 1 continued



Minimize $C=30x+20y$

x	y	$C=30x+20y$
0	88	1760
20	56	1720
50	32	2140
130	0	3900

the farmer should buy 20 bags of A and 56 bags of B to minimize cost at $1720

5.2/5.3 Maximization and Minimization Applications

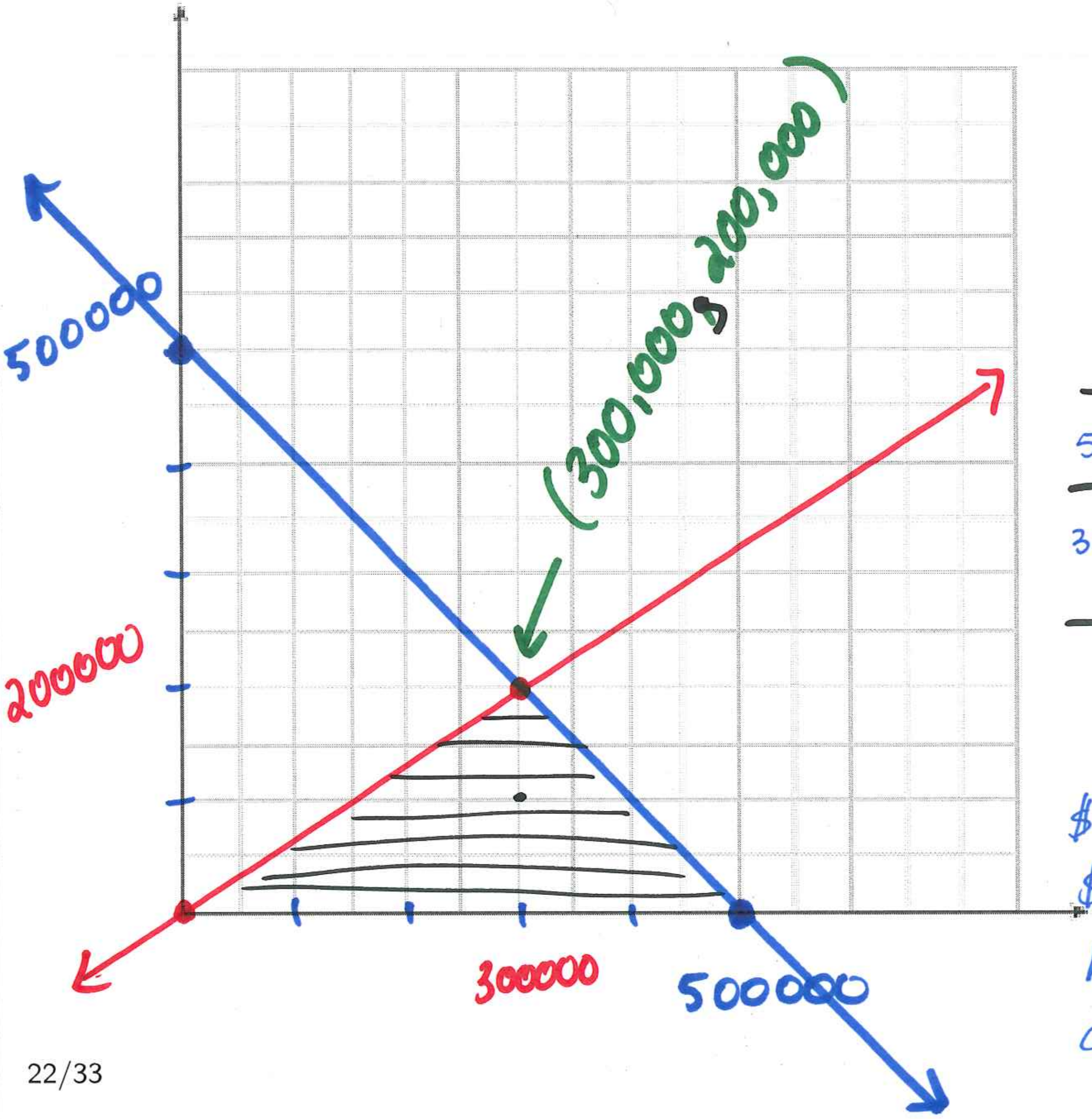
Example 2: A financier plans to invest up to \$500000 in two projects. Project A yields a return of 10% on the investment, whereas Project B yields a return of 15%. Because Project B is riskier than Project A, the financier has decided that the investment in Project B should not exceed 40% of the total investment. How much should she invest in each project if she wishes to maximize the total returns on her investments? What is the maximum return?

x = amount invested in project A

y = amount invested in project B

Objective: Maximize $R = .10x + .15y$

Example 2 continued



$y \leq -x + 500000$

$y \leq \frac{2}{3}x$

x	y	R=.1x+.15y
0	0	0
500000	0	50000
300000	200000	60000

The financer should invest \$300.000 in project A and \$200000 in project B to maximize the return at \$60,000

Example 2 continued

Constraints:

$x \geq 0, y \geq 0$

		x-int	y-int
● Amount invested	$x+y \leq 500000$	$(500000,0)$	$(0,500000)$
● Red underlined	$-.4x+.6y \leq 0$	$(0,0)$	$(0,0)$

Amount in project B ≤ 40% of the total investment

$$y \leq .4(x+y)$$

$$y \leq .4x+.4y$$

$$y-.4y-.4x \leq 0$$

$$-.4x+.6y \leq 0$$

$$-.4x+.6y=0$$

$$\Rightarrow \quad .6y=.4x$$

$$y=\frac{.4}{.6}x$$

$$\boxed{y=\frac{2}{3}x}$$

Theorem: Existence of Solutions to LP Problems

Suppose we are given a linear programming problem with a feasible set S and an objective function $P = ax + by$.

- If S is bounded, then P has both a maximum and a minimum value on S. bounded → draw a circle around S
- If S is unbounded and both a and b are non-negative, then P has a minimum value on S provided that the constraints defining S include the inequalities $\underline{x \geq 0}$ and $\underline{y \geq 0}$.
- If S is the empty set, then the linear programming problem has no solution. That is, P has neither a maximum nor a minimum value.

5.3 Minimization Problems

Example 1: A diet is to contain at least 2400 units of vitamins, 1800 units of minerals, and 1200 calories. Two foods, Food A and Food B are to be purchased. Each unit of Food A provides 50 units of vitamins, 30 units of minerals, and 10 calories. Each unit of Food B provides 20 units of vitamins, 20 units of minerals, and 40 calories. If Food A costs $2 per unit and Food B cost $1 per unit, how many units of each food should be purchased to keep costs at a minimum?

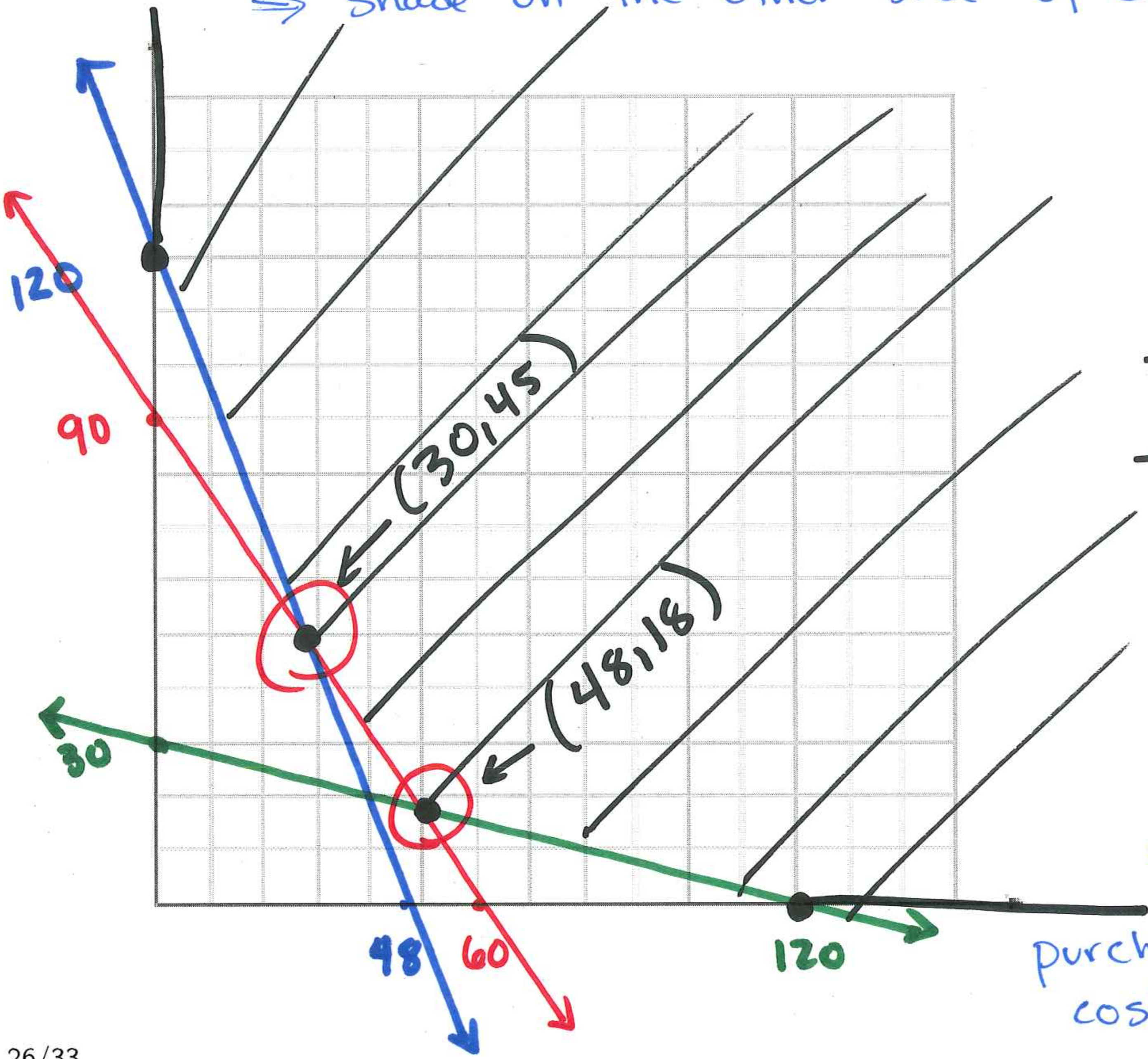
x = # of units of food A purchased

y = # of units of foob B purchased

Objective: $C = 2x + y \leftarrow$ minimize

Example 1 continued

(0,0) does not satisfy any of the constraints
⇒ shade on the other side of each line



x	y	C = 2x+y
0	120	120
120	0	240
48	18	114
30	45	105

30 units of food A and 45 units of food B should be purchased to minimize cost at $105

Example 1 continued

Constraints:

$x \geq 0 \;, y \geq 0$

	x-int	y-int
$50x + 20y \geq 2400$	$(48, 0)$	$(0, 120)$
$30x + 20y \geq 1800$	$(60, 0)$	$(0, 90)$
$10x + 40y \geq 1200$	$(120, 0)$	$(0, 30)$

Solve:

$$\begin{array}{rr} & 50x + 20y = 2400 \\ - & 30x + 20y = 1800 \\ \hline & 20x = 600 \end{array}$$

$$x = 30$$

$$y = 45$$

Solve:

$$30x + 20y = 1800$$

$$(-3) \cdot (10x + 40y = 1200)$$

$$\Downarrow$$

$$\begin{array}{rr} & 30x + 20y = 1800 \\ + & -30x - 120y = -3600 \\ \hline & -100y = -1800 \end{array}$$

$$\boxed{y = 18}$$

$$10x + 40(18) = 1200$$

$$10x = 480 \Rightarrow \boxed{x = 48}$$

5.3 Minimization Problems

Example 3:A professor gives two types of quizzes, objective and recall. He is planning to give at least 15 quizzes this quarter. The student preparation time for an objective quiz is 15 minutes and for a recall quiz 30 minutes. The professor would like a student to spend at least 5 hours (300 minutes) preparing for these quizzes above and beyond the normal study time. The average score on an objective quiz is 7, and on a recall type 5, and the professor would like the students to score at least 85 points on all quizzes. It takes the professor one minute to grade an objective quiz, and 1.5 minutes to grade a recall type quiz. How many of each type should he give in order to minimize his grading time?

- x = # of objective quizzes given
  y = # of recall quizzes given

Objective : Minimize $G = 1 \cdot x + 1.5y$

Example 3 continued

Constraints: $x \geq 0, y \geq 0$

	y-int	x-int
$x + y \geq 15$	(0, 15)	(15, 0)
$15x + 30y \geq 300$	(0, 10)	(20, 0)
$7x + 5y \geq 85$	(0, 17)	(12.14, 0)

↑ 85/7

red/green

$x + y = 15$

$7x + 5y = 85$

(5, 10)

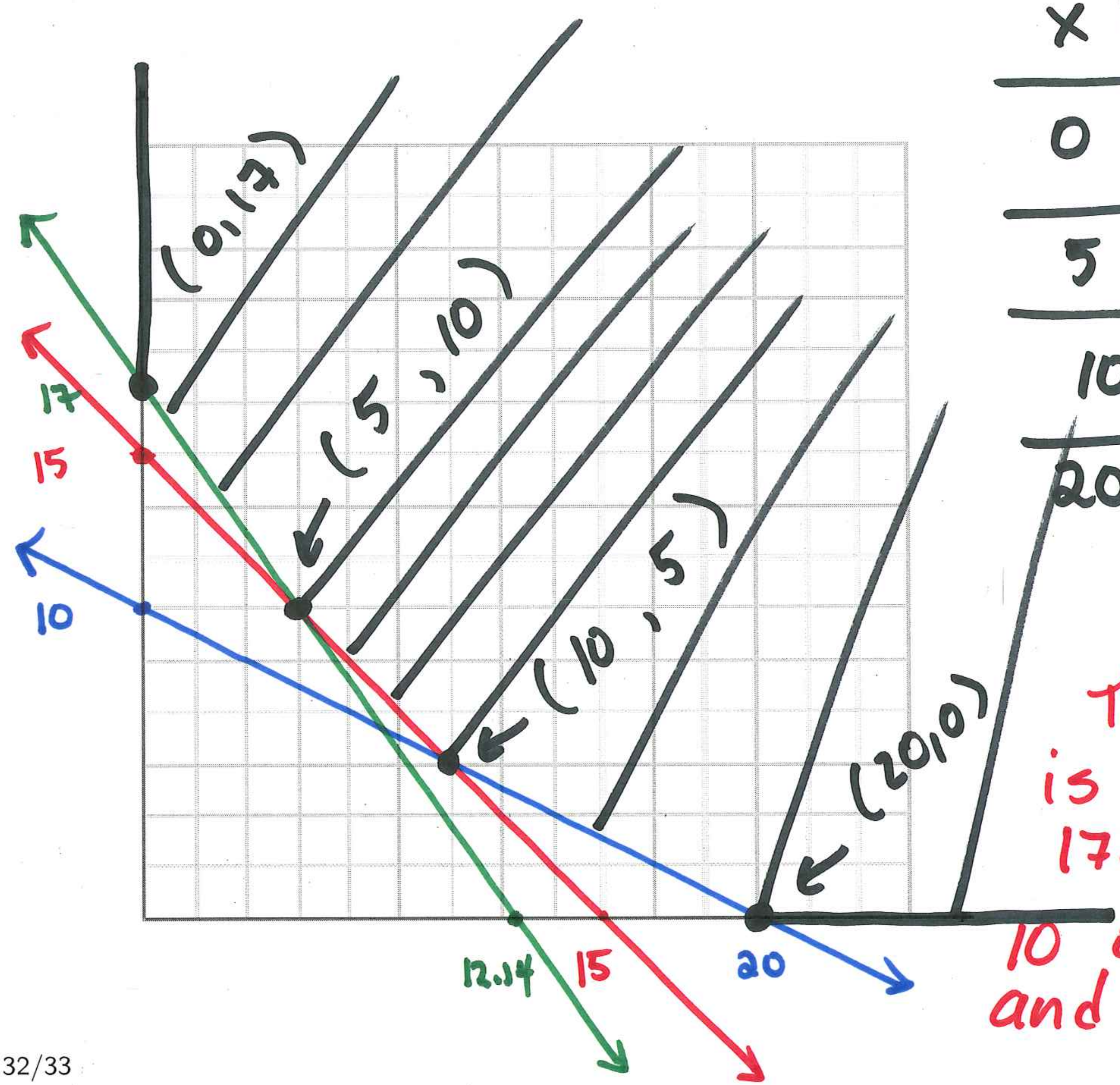
red/blue

$x + y = 15$

$15x + 30y = 300$

(10, 5)

Example 3 continued



x	y	G=x+1.5y
0	17	25.5
5	10	20
10	5	17.5
20	0	20

The grading time is minimized at 17.5 minutes for 10 objective quizzes and 5 recall quizzes.

7.2 Maximization By The Simplex Method

In chapter 5, we used the corner point method to solve linear programming problems. The geometric approach will not work for problems that have more than two variables.

Real life problems consist of thousands of variables and constraints. Although it is still possible to solve these problems geometrically, it would be tedious and time consuming.

We need another method to solve linear programming problems that doesn't require finding corner points. The method must also be simple enough that we don't have to evaluate the objective function at each corner point.

This simpler method is algebraic and is called **the simplex method**.

7.2 Standard Linear Programming Problems

A **standard maximization problem** is one in which

- The objective function is to be maximized.
- All of the variables involved are non-negative.
- All other linear constraints may be written so that the expression involving the variables is less than or equal to a non-negative constant.

Bad! $0 \leq .4x - .6y$

Good! $-.4x + .6y \leq 0$

7.2 Slack Variables

A part of the simplex method that will be discussed today is the introduction of **slack variables**.

- A **slack variable** is used to change an inequality into an equality.
- Suppose $x + y \leq 50$ with $x \geq 0$ and $y \geq 0$.
- We can replace this inequality with the equality

$$x + y + u = 50 \quad \text{where} \quad x \geq 0, y \geq 0, u \geq 0.$$

- u is called the slack variable.
- Slack variable usually represent the leftover amount of a resource.

7.2 Setting up the Simplex Table

Example 1: National Business Machines Corporation manufactures two models of printers: *A* and *B*. Each model *A* costs \$100 to make and each model *B* costs \$150. The profits are \$30 for each model *A* and \$40 for each model *B* printer. If the total number of printers demanded each month does not exceed 2500 and the company has earmarked no more than \$600000 per month for manufacturing costs, find how many units of each model should be made each month to maximize monthly profit. What is the largest monthly profit?

$x+y \le 2500$

Objective

7.2 Example 1 continued

- Let x be the number of units of model A produced.
- Let y be the number of units of model B produced.

$$\left.\begin{array}{l}\bullet\ \text{Printers made: } x+y \leq 2500 \\[2ex] \bullet\ \text{Costs: } 100x+150y \leq 600000 \\[2ex] \bullet\ \text{Non-negativity: } x \geq 0, y \geq 0\end{array}\right\}\ \text{Constraints}$$

- Objective: Maximize $P = 30x + 40y$

7.2 Example 1 continued

- slack Let u be the number of printers less than 2500 made.
- Slack Let v be the amount of money not spent from the budget.
- Row 1 Printers made: $x + y + u = 2500$
- Row 2 Costs: $100x + 150y + v = 600000$
- Non-negativity: $x \geq 0, y \geq 0, u \geq 0, v \geq 0$
- Row 3 Objective: Maximize $P = 30x + 40y$

Always the bottom row !!

7.2 Example 1 continued

- Put the the equations into the following table

$x=0$
$y=0$

$u=2500$
$v=600000$
$P=0$

- $$\left[\begin{array}{cc|ccc|c} x & y & u & v & P & RHS \\ \hline 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ \hline -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

- The bottom row is the objective function rewritten as

$$-30x - 40y + P = 0$$

- Right now, x and y are called **non-basic variables** and u, v and P are call **basic variables**.

7.2 Example 1 continued

- The corner points of the feasible region correspond to letting the non-basic variable equal 0.
- Right now, we let $x = 0$ and $y = 0$ which tells us that $u = 2500$, $v = 600000$ and $P = 0$.
- This means that if we make zero units of model A and zero units of model B, then there are 2500 printers not made, \$600000 of unused money from the budget, and \$0 profit.

7.2 Example 1 continued

- Next we want to switch which variables are basic and non-basic in a way that increases profit.
- How to choose which columns to switch: Look for the largest negative number in the bottom row of the simples table (if all entries are positive, then you are done).

Make this a unit column

- $$\left[\begin{array}{ccccc|c} x & y & u & v & P & RHS \\ \hline 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ \hline -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$
- In this case will will use column 2 because this causes a larger increase in profit.

7.2 Example 1 continued

- How do we choose which row to pivot about? Look at each row with a positive entry in the chosen column.
- For each row, divide the right-hand side entry by the entry in the chosen column. The row with the smallest ratio is the row to choose.
- $$\text{pivot} \longrightarrow \left[\begin{array}{ccccc|c} x & y & u & v & P & RHS \\ \hline 1 & \boxed{1} & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ \hline -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

Ratio

$2500/1 = \boxed{2500}$

$600000/150 = 4000$

- In this problem, we have $\frac{2500}{1} = 2500$ and $\frac{600000}{150} = 4000$. We choose to pivot in the first row because this is the smaller ratio.

↑ positive

7.2 Example 1 continued

- Next we want to switch which variables are basic and non-basic in a way that increases profit.
- How to choose which columns to switch: Look for the largest negative number in the bottom row of the simples table (if all entries are positive, then you are done).

Make this a unit column

- $$\left[\begin{array}{ccccc|c} x & y & u & v & P & RHS \\ \hline 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ \hline -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

$x+y \leq 2500$

$100x+150y \leq 600000$

$P = 30x+40y$

- In this case will will use column 2 because this causes a larger increase in profit.

7.2 Example 1 continued

- How do we choose which row to pivot about? Look at each row with a positive entry in the chosen column.
- For each row, divide the right-hand side entry by the entry in the chosen column. The row with the smallest ratio is the row to choose.
- pivot

$$\left[\begin{array}{ccccc|c} x & y & u & v & P & RHS \\ \hline 1 & \boxed{1} & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ \hline -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

Ratio

$2500/1 = \boxed{2500}$

$600000/150 = 4000$

- In this problem, we have $\frac{2500}{1} = 2500$ and $\frac{600000}{150} = 4000$. We choose to pivot in the first row because this is the smaller ratio.

↑ positive

7.2 Example 1 continued

- Pivot about the entry in Row 1, Column 2.

make zero

$$\left[\begin{array}{ccccc|c} x & y & u & v & P & RHS \\ \hline 1 & 1 & 1 & 0 & 0 & 2500 \\ 100 & 150 & 0 & 1 & 0 & 600000 \\ \hline -30 & -40 & 0 & 0 & 1 & 0 \end{array}\right]$$

unit columns

$$\begin{array}{rcl} R_2 & \rightarrow & R_2 - 150R_1 \\ R_3 & \rightarrow & R_3 + 40R_1 \end{array} \quad \left[\begin{array}{ccccc|c} x & y & u & v & P & RHS \\ \hline 1 & 1 & 1 & 0 & 0 & 2500 \\ -50 & 0 & -150 & 1 & 0 & 225000 \\ \hline 10 & 0 & 40 & 0 & 1 & 100000 \end{array}\right]$$

- Now we have x and u as the non-basic variables so we have $x = 0, y = 2500, u = 0, v = 225000$, and $P = 100000$.
- Since there are no negative numbers left in the bottom row, the simplex method is complete and the solution is optimal.
- The maximum profit of \$100000 occurs when 2500 model B printers and 0 model A printers are made. There is \$225000 unspent from the budget.

7.2 The Simplex Algorithm

1. Set up the initial simplex table
2. Determine whether the optimal solution has been reached by examining all the entries in the last row to the left of the vertical line.
    - If all the entries are non-negative, the optimal solution has been reached. Proceed to step 4.
    - If there are one or more negative entries, the optimal solutions has not been reached. Proceed to step 3.
3. Perform the pivot operation. Locate the pivot element and convert that column to a unit column. Return to step 2.
4. Determine the optimal solution(s). Non-basic variables get set equal to zero and the other variables are read off the final table.

7.2 Maximization by the Simplex Method

Row 3

Example 2: Maximize $P = 5x + 3y$ subject to $\Rightarrow -5x - 3y + P = 0$

$$\begin{cases} x + y \le 80 \Rightarrow & x + y + u = 80 \quad \text{Row1} \\ 3x \le 90 \Longrightarrow & 3x + v = 90 \quad \text{Row 2} \end{cases}$$

and $x \geq 0, y \geq 0$.

$x \ge 0, y \ge 0, u \ge 0, v \ge 0$

Pivot column

x	y	u	v	P	constant	Ratio
1	1	1	0	0	80	80/1=80
3	0	0	1	0	90	90/3=30
-5	-3	0	0	1	0	

pivot entry

7.2 Example 2 continued

$$R_2 \longrightarrow R_2 \cdot \frac{1}{3}$$

$$\longrightarrow$$

$$\begin{array}{ccccc|c} x & y & u & v & P & \text{RHS} \\ \hline \boxed{1} & 1 & 1 & 0 & 0 & 80 \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 30 \\ \hline \boxed{-5} & -3 & 0 & 0 & 1 & 0 \end{array}$$

make zero

$$R_1 \longrightarrow R_1 - R_2$$

$$\longrightarrow$$

$$R_3 \longrightarrow R_3 + 5R_2$$

Pivot column

$$\begin{array}{ccccc|c|c} x & y & u & v & P & \text{RHS} & \text{Ratio} \\ \hline 0 & \boxed{1} & 1 & -\frac{1}{3} & 0 & 50 & 50/1=50 \\ 1 & 0 & 0 & \frac{1}{3} & 0 & 30 & 30/0=\text{DNE} \\ \hline 0 & -3 & 0 & \frac{5}{3} & 1 & 150 & \end{array}$$

pivot entry

7.2 Example 2 continued

$R_3 \to R_3 + 3R_1$

$$\begin{array}{ccccc|c} x & y & u & v & P & \text{RHS} \\ \hline 0 & 1 & 1 & -1/3 & 0 & 50 \\ 1 & 0 & 0 & 1/3 & 0 & 30 \\ \hline 0 & 0 & 3 & 2/3 & 1 & 300 \end{array}$$

unit columns

$x = 30$, $y = 50$, $P = 300$

$u = 0$, $v = 0$ ← non-unit columns

x, y, P are basic variables
u, v are non-basic variables

P is maximized at 300 when $(x, y) = (30, 50)$

7.2 Maximization by the Simplex Method

Example 4: Boise Lumber manufactures prefabricated houses. They offer three models; standard, deluxe, and luxury. Each house is prefabricated and partially assembled in a factory. The final assembly is done on-site.

The dollar amount of building material required, the amount of prefabrication and on-site labor, and the profit per unit are given in the following table.

		x *Standard*	y *Deluxe*	z *Luxury*
constraints	Material	\$6000	\$8000	\$10000
	Factory Labor	240	220	200
	On-site Labor	180	210	300
Objective →	Profit	\$3400	\$4000	\$5000

They have \$8200000 budgeted for materials, 21800 hours of factory labor, and 237000 hours of on-site labor. How many of each house should be built to maximize profit?

7.2 Example 4 continued

Let $x =$ the number of standard houses built.
Let $y =$ the number of deluxe houses built.
Let $z =$ the number of luxury houses built.

Objective: Maximize

$$P = 3400x + 4000y + 5000z$$

$$-3400x - 4000y - 5000z + P = 0$$

Constraints:

Slack

$$\begin{aligned} \text{Material:} \quad & 6000x + 8000y + 10000z \leq 8200000 \leftarrow u \\ \text{Factory Labor:} \quad & 240x + 220y + 200z \leq 218000 \leftarrow v \\ \text{On-Site Labor:} \quad & 180x + 210y + 300z \leq 237000 \leftarrow w \\ & x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

7.2 Example 4 continued

No basic

Basic

$$\begin{array}{ccc|cccc|c|c}
x & y & z & u & v & w & p & \text{Constant} & \text{Ratio} \\ \hline
6000 & 8000 & 10000 & 1 & 0 & 0 & 0 & 8200000 & \frac{8200000}{10000} \\
240 & 220 & 200 & 0 & 1 & 0 & 0 & 218000 & 218000/200 \\
180 & 210 & 300 & 0 & 0 & 1 & 0 & 237000 & 237000/300 \\ \hline
-3400 & -4000 & -5000 & 0 & 0 & 0 & 1 & 0 &
\end{array}$$

Pivot

smallest positive ratio

u = \$ left over

V = # of leftover factor hours

W = # of left over on-site hours

$(x, y, z) = (0, 0, 0)$

$(u, v, w) = (8200000, 218000, 237000)$

Apply the following row operations;

$R_3 \rightarrow R_3(\frac{1}{300})$

$R_1 \rightarrow R_1 - 10000R_3$

$R_2 \rightarrow R_2 - 200R_3$

$R_4 \rightarrow R_4 + 5000R_3$

Non-basic

Basic

smallest positive

x	y	z	u	v	w	P	Constant	Ratios
0	1000	0	1	0	$-100/3$	0	300000	300 000/1000
120	80	0	0	1	$-2/3$	0	60000	60000/80
$3/5$	$7/10$	1	0	0	$1/300$	0	790	790/(7/10)
-400	-500	0	0	0	$50/3$	1	3950000	

pivot

Now pivot in the y column, first row.

Note: At this step in the simplex method, $(x, y, z) = (0, 0, 790)$ and $(u, v, w) = (300000, 60000, 0)$.

Apply the following row operations;

$R_1 \rightarrow R_1(\frac{1}{1000})$

$R_2 \rightarrow R_2 - 80R_1$

$R_3 \rightarrow R_3 - \frac{7}{10}R_1$

$R_4 \rightarrow R_4 + 500R_1$

Non-basic

Basic

smallest positive

x	y	z	u	v	w	P	Constant	Ratios
0	1	0	1/1000	0	−1/30	0	300	300/0 = DNE
120	0	0	−2/25	1	2	0	36000	36000/120
3/5	0	1	0	0	2/75	0	580	580/(3/5)
−400	0	0	1/2	0	0	1	4100000	

pivot

Now pivot in the x column, second row.

(0, 300, 580)

Note: At this step in the simplex method, $(x, y, z) =$ [illegible] and $(u, v, w) = (0, 36000, 0)$.

Apply the following row operations;

$R_2 \rightarrow R_2(\frac{1}{120})$

$R_3 \rightarrow R_3 - \frac{3}{5}R_2$

$R_4 \rightarrow R_4 + 400R_2$

Non-basic

Basic

x	y	z	u =0	v =0	w =0	P	Constant	
0	1	0	1/1000	0	−1/30	0	300	— y
1	0	0	0	1/120	1/60	0	300	— x
0	0	1	0	−1/200	1/60	0	400	— z
0	0	0	7/30	10/3	20/3	1	4220000	

The simplex method is done now that there are no negatives in the bottom row.

The company will maximize their profit at \$4220000 when $(x, y, z) = (300, 300, 400)$. Also, $(u, v, w) = (0, 0, 0)$ so all the resources have been used.

7.3 Minimization by the Simplex Method

In this section we will solve standard linear programming minimization problems by the simplex method.

The type of minimization problem we will consider requires that all the inequalities are of the form $ax + by \geq c$ where c is any constant (Notice that we don't require c to be non-negative as in the maximization problems). Also, all variables involved must be non-negative.

The procedure to solve these problems involves converting the minimization problem into its **dual** maximization problem. Once converted to a maximization problem, we can use the same simplex algorithm from 7.2 to find the solution.

7.3 Converting a Minimization Problem to its Dual

Example 1: Convert the following minimization problem to its dual.

Minimize: $C = 3x + 5y$ Row 3

Subject to: $x \geq 0,\ y \geq 0$

$4x + 3y \geq 20$ Row 1

$7x + 5y \geq 40$ Row 2

Step 1: Convert into a maximization problem

Create a table containing the info

x	y	
4	3	20
7	5	40
3	5	0

$\xrightarrow[\text{dualize}]{}$

4	7	3
3	5	5
20	40	0

7.3 Converting a Minimization Problem to its Dual

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Subject to: $x \geq 0,\ y \geq 0$

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$7x + 5y \geq 40$ Row 2

Step 1: Convert into a maximization problem

Create a table containing the info

$$\begin{array}{c|cc|c} & x & y & \\ u & 4 & 3 & 20 \\ v & 7 & 5 & 40 \\ \hline & 3 & 5 & 0 \end{array} \xrightarrow[\text{dualize}]{} \begin{array}{c|cc|c} & u & v & \\ x & 4 & 7 & 3 \\ y & 3 & 5 & 5 \\ \hline & 20 & 40 & 0 \end{array}$$

7.3 Example 1 continued

Use the usual slack variables in the objective and constraints for the dual maximization problem.

Objective: $P = 20u + 40v \Rightarrow -20u - 40v + P = 0$

Constraints: $4u + 7v \le 3 \Rightarrow 4u + 7v + x = 3$

$3u + 5v \le 5 \Rightarrow 3u + 5v + y = 5$

$u, v \ge 0$

$x, y \ge 0$

u	v	x	y	P	RHS
4	7	1	0	0	3
3	5	0	1	0	5
-20	-40	0	0	1	0

7.3 The Duality Principle

The duality principle says that the objective function of the minimization problem reaches its minimum if and only if the objective function of its dual reaches its maximum. And when they do, they are equal.

Example 2: Minimize $C = 2x + 5y$

Subject to: $x, y \geq 0$

$$x + 2y \geq 6$$

$$3x + 2y \geq 6$$

Dualize:

$$\begin{array}{ccc|c} & x & y & \\ u & 1 & 2 & 6 \\ v & 3 & 2 & 6 \\ \hline & 2 & 5 & 0 \end{array} \longrightarrow \begin{array}{ccc|c} & u & v & \\ x & 1 & 3 & 2 \\ y & 2 & 2 & 5 \\ \hline & 6 & 6 & 0 \end{array}$$

7.3 Example 2 continued

Objective: $P = 6u + 6v$

Subject to: $u + 3v \leq 2 \Rightarrow u + 3v + x = 2$

$2u + 2v \leq 5 \Rightarrow 2u + 2v + y = 5$

$u, v \geq 0$

Create Simplex Table:

u	v	x	y	P	RHS	Ratios
1	3	1	0	0	2	$2/1 = 2$
2	2	0	1	0	5	$5/2 = 2.5$
−6	−6	0	0	1	0	

Pivot entry

7.3 Example 2 continued

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 + 6R_1$

$$\begin{array}{ccccc|c}
u & v & x & y & P & RHS \\
\hline
1 & 3 & 1 & 0 & 0 & 2 \\
0 & -4 & -2 & 1 & 0 & 1 \\
\hline
0 & 12 & \multicolumn{2}{c}{\boxed{6 \quad\quad 0}} & 1 & 12
\end{array}$$

Interpret the final table in a special way to find the solution to the minimization. Read the solution from the last row.

C is minimized at $(x, y) = (6, 0)$ for a value of 12.

7.3 Minimization by the Simplex Method

1. Set up the problem.
2. Write a matrix whose rows represent each constraint with the objective function in the bottom row.
3. Write the transpose of the matrix by interchanging the rows and columns.
4. Write the dual maximization problem associated to the transpose from step 3.
5. Solve the dual problem by the simplex method.
6. The optimal solution is found in the bottom row of the final matrix in the columns corresponding to the slack variables and the minimum value of the objective function is the same as the maximum value of the dual.

7.3 Minimization by the Simplex Method

Example 3: Minimize $C = 4x + 6y + 7z$

Subject to: $x, y, z \geq 0$

$$x + y + 2z \geq 20$$

$$x + 2y + z \geq 30$$

$$\begin{array}{cccc|c} & x & y & z & \\ u & 1 & 1 & 2 & 20 \\ v & 1 & 2 & 1 & 30 \\ \cline{2-5} & 4 & 6 & 7 & 0 \end{array} \xrightarrow{\text{dualize}} \begin{array}{ccc|c} & u & v & \\ x & 1 & 1 & 4 \\ y & 1 & 2 & 6 \\ z & 2 & 1 & 7 \\ \cline{2-4} & 20 & 30 & 0 \end{array}$$

7.3 Example 3 continued

Maximize:

$$P = 20u + 30v \Rightarrow -20u - 30v + P = 0$$

Subject to:

$$u + v \le 4 \longrightarrow u + v + x = 4$$

$$u + 2v \le 6 \longrightarrow u + 2v + y = 6$$

$$2u + v \le 7 \longrightarrow 2u + v + z = 7$$

$$u, v \ge 0 \qquad x, y, z \ge 0$$

u	v	x	y	z	P	RHS	Ratio
1	1	1	0	0	0	4	4/1 = 4
1	2	0	1	0	0	6	6/2 = 3
2	1	0	0	1	0	7	7/1 = 7
−20	−30	0	0	0	1	0	

Pivot entry

7.3 Example 3 continued

$R_2 \to R_2 \cdot \frac{1}{2}$

$R_1 \to R_1 - R_2$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 + 30R_2$

oops

u	v	x	y	z	P	RHS	Ratios
$\boxed{\frac{1}{2}}$	0	1	$-\frac{1}{2}$	0	0	1	2
$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	0	3	6
$\frac{3}{2}$	0	0	$-\frac{1}{2}$	1	0	4	8/3
-5	0	0	15	0	1	90	

pivot entry

7.3 Example 3 continued

$R_1 \rightarrow R_1 \cdot 2$

$R_2 \rightarrow R_2 - \frac{1}{2} R_1$

$R_3 \rightarrow R_3 - \frac{3}{2} R_1$

$R_4 \rightarrow R_4 + 5 R_1$

u	v	x	y	z	P	RHS
1	0	2	-1	0	0	2
0	1	-1	1	0	0	2
0	0	-3	1	1	0	1
0	0	10	10	0	1	100

C has a minimum of 100

at $(x, y, z) = (10, 10, 0)$