9.2 Simple Interest and Discount

Simple Interest:

- It costs money to borrow money. The cost one pays for the use of money is called **interest**.
- The money being borrowed or loaned is called the **principal or present value**.
- When interest is only paid on the original amount borrowed, it is called **simple interest**.
- The interest is charged for the amount of time the money is borrowed. The interest rate is set by the lender and is often specified for a year, it may be specified for a week, an month, or a quarter.

Simple Interest

If an amount P is borrowed for a time t at an interest rate of r per time period, then the interest is given by

$$I = P \cdot r \cdot t$$

The total amount of the transaction A is called the **accumulated value** or the **future value** and is the sum of the principal and interest.

$$A = P + I = P + Prt$$

or

$$A = P(1 + rt)$$

where the rate r is expressed as a decimal.

Simple Interest

Example 1: Sam borrows \$800 for 7 months at a simple interest rate of 7% per year. Find the interest and the accumulated value.

$$I = 800\left(\underset{\uparrow \text{ per year}}{.07}\right)\left(\underset{\uparrow \text{ \# of years}}{\tfrac{7}{12}}\right) = \$32.67$$

$$FV = A = P + I = 800 + 32.67 = \$832.67$$

$$\parallel$$

-or-

$$A = P(1+rt) = 800\left(1 + (.07)\left(\tfrac{7}{12}\right)\right)$$

Example 2: Tina deposits \$2000 into and account that pays 7% simple interest per year. How much money will be in the account after 5 years? How much did Tina earn in interest?

$$FV = A = P(1+rt) = 2000(1+.07(5))$$
$$= \$2700$$

$$I = A - P = 2700 - 2000 = \$700$$

Example 3: Steve owes an accumulated value of \$5500 which includes the 7% annual simple interest over the 4 years he has borrowed the money. How much did Steve originally borrow?

$$A = P(1 + rt)$$

$$5500 = P(1 + (.07)(4))$$

$$5500 = P(1.28)$$

$$P = \frac{5500}{1.28} = \$4296.88$$

P

Example 4: Find the length of a loan in months if \$300 is borrowed with an annual simple interest rate of 7% and \$345 is repaid at the end of the loan.

A

APR

$$A = P(1 + rt)$$

$$345 = 300(1 + (.07)t)$$

$$\frac{345}{300} = 1 + .07t$$

$$1.15 - 1 = .07t$$

$$.15 = .07t$$

$$t = \frac{.15}{.07} \approx 2.1429 \text{ years} \Rightarrow t \approx 25.71 \text{ months}$$

Example 5: Find the annual simple interest rate of a loan where \$400 is borrowed and \$430 is repaid at the end of 15 months.

P   A

$15/12 = t$

$$A = P(1 + rt)$$

$$430 = 400\left(1 + r\left(\tfrac{15}{12}\right)\right)$$

$$\frac{430}{400} = 1 + \frac{15}{12}r$$

$$\frac{12}{15}(1.075 - 1) = \left(\frac{\cancel{15}}{\cancel{12}}r\right)\frac{\cancel{12}}{\cancel{15}}$$

$$.06 = r = 6\%$$ annual simple interest

Discounts and Proceeds

Discounts: Banks often deduct the simple interest from the loan amount at the time the loan is made. When this happens, we say the loan has been discounted.

Proceeds: The interest that is deducted is called the discount and the actual amount given to the borrower is called the proceeds.

The amount the borrower is obligated to repay is called the **maturity value**.

Discounts and Proceeds

If an amount M is borrowed for a time t at a discount rate of r per year, then the discount is

$$D = M \cdot r \cdot t$$

The proceeds P, the actual amount the borrower gets, is given by

$$P = M - D = M - Mrt$$

or

$$P = M(1 - rt)$$

Where r is the interest rate expressed as a decimal.

M $t = \frac{18}{12}$

Example 6: Helen borrows \$1000 for 18 months at a simple interest rate of 12% per year. Find the discount and proceeds.

$r = .12$

$$D = M \cdot r \cdot t = 1000(.12)\left(\frac{18}{12}\right)$$
$$= \$180$$

$$P = M - D = 1000 - 180 = \$820$$

$r = .12$ P $t = \frac{18}{12}$

Example 7: Helen wants to receive \$1000 for 18 months at a simple interest rate of 12% per year. What amount of loan should she apply for?

$$P = M(1 - rt)$$

$$1000 = M\left(1 - (.12)\left(\frac{18}{12}\right)\right)$$

$$1000 = M(.82)$$

$$\frac{1000}{.82} = \$1219.51$$

Accumulation and Discount Factors

- To convert an earlier time to a later time, you multiply by an *accumulation factor*.

- To convert a later time to an earlier time, you multiply by a *discount factor*.

- Accumulation factors and discount factors are reciprocals of each other.

Discount Factor

Example 8: You are to receive \$1000 one year from now, and another \$1500 three years from now. How much is this money worth right now? Assume that money increases in value by 3% every year.

Now | 1 year | 3year

$1000(1.03)^{-1} \longleftarrow 1000$

$= 970.87$

$1500(1.03)^{-3} \longleftarrow 1500$

$= 1372.71$

Total = \$2343.58

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$1500(1.03)^{-3} \longleftarrow 1500$

$= 1372.71$

Total = \$2343.58

Example 9: You are to receive \$1000 one year from now, and another \$1500 three years from now. How much is this money worth three years from now? Assume that money increases in value by 3% every year.

Now

1 year

3 year

$1000 \longrightarrow 1000(1+.03)^2$

$= 1060.90$

$+ \quad 1500$

\$2560.90

Accumulation and discount factors allow us to push the value of money forward and backward in time.

Suppose you invest \$600 into an account that earns 4% simple annual interest. In 3 years, the value of your investment is

$$A = 600(1 + .04)(1 + .04)(1 + .04) = 600(1.04)^3$$

For each year you keep the money in the account, you investment increases by an accumulation factor of 1.04 which is 104% of the prior year's worth.

9.3 Compound Interest

Now suppose you invest \$600 in an account that earns 4% simple annual interest. However, the interest is applied to your investment biannually (2 times a year).

Every six months, half of 4% interest will be applied to your investment. So after 3 years, the value of your money will be

$$A = 600\left(1+\frac{.04}{2}\right)\left(1+\frac{.04}{2}\right)\left(1+\frac{.04}{2}\right)\left(1+\frac{.04}{2}\right)\left(1+\frac{.04}{2}\right)\left(1+\frac{.04}{2}\right)$$

$$= 600\left(1+\frac{.04}{2}\right)^{6} = 600\left(1+\frac{.04}{2}\right)^{2\cdot 3}$$

In general, if we invest a lump-sum amount of P dollars at an annual interest rate r, compounded n times per year, then after t years the accumulated value A is

$$FV = A = P\left(1 + \frac{r}{n}\right)^{nt}$$

P t r%

Example 1: How much is $2000 worth in 4 years if it earns 6.25% annual interest compounded weekly? ($\underset{n}{\underline{52}}$ weeks in a year)

$$A = 2000\left(1 + \frac{.0625}{52}\right)^{52(4)}$$

(calculator) $\approx$ \$ 2567.67

Example 2: How much should be invested in an account earning
r 9% annual interest compounded daily so that the value of the
account is $3000 in four years? 365

FV=A t

$$3000 = P\left(1+\frac{.09}{365}\right)^{365(4)}$$

$$3000 = P(1.43326581)$$

$$P = \$\,2093.12$$

$n=12$

Example 3: If you make a deposit into an account, at what annual interest rate compounded monthly should you invest if you would like to double your investment in 18 months?

$t=1.5$

$2P$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$2\cancel{P} = \cancel{P}\left(1+\frac{r}{12}\right)^{12(1.5)}$$

$$\sqrt[m]{x^n} = x^{n/m}$$

$$(2)^{1/18} = \left(\left(1+\frac{r}{12}\right)^{18}\right)^{1/18}$$

$$1.039259 = 1+\frac{r}{12}$$

$$.039259 = \frac{r}{12}$$

$$.47111 \approx r$$

$$r \approx 47.11\%$$

$r\%$ $\qquad A = 2P$

Example 4: How long does it take for an investment to double at an annual interest rate of 4% compounded quarterly? $\leftarrow n = 4$

$\ln(x^p) = p\ln(x)$

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$2\cancel{P} = \cancel{P}\left(1+\frac{.04}{4}\right)^{4t}$$

$$\ln(2) = \ln\left((1.01)^{4t}\right)$$

$$\ln(2) = 4t \cdot \ln(1.01)$$

$$\frac{\ln(2)}{4\ln(1.01)} = t \quad \Rightarrow \quad t \approx 17.42 \text{ years}$$

9.3 Continuous Compounding

When interest is compounded "infinitely many times" per year, we say that the interest is compounded continuously.

In the table below, we can see what happens to \$1 at a annual rate of 100% interest compounded at different time periods.

Frequency of Compounds	Formula	Value After 1 year
Annually	$1(1+1)$	\$2
Semiannually	$1\left(1+\frac{1}{2}\right)^{2}$	\$2.25
Quarterly	$1\left(1+\frac{1}{4}\right)^{4}$	\$2.44140625
Monthly	$1\left(1+\frac{1}{12}\right)^{12}$	\$2.61303529
Daily	$1\left(1+\frac{1}{365}\right)^{365}$	\$2.71456748
Hourly	$1\left(1+\frac{1}{8760}\right)^{8760}$	\$2.71812699

The value of \$1 earning 100% annual interest compounded continuously is $\$2.718281828\ldots = e$.

9.3 Continuous Compounding

As the number of compounds become sufficiently large, the expression

$$\left(1+\frac{1}{n}\right)^{n}$$

approaches the value $e \approx 2.71828$.

Therefore, the accumulated value

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

approaches

$$A = Pe^{rt}$$

P r%

Example 5: If $3000 is invested at 9% annual interest compounded continuously, what is the future value in four years?

t=4

$$A = Pe^{rt}$$

$$FV = A = 3000e^{(.09)(4)}$$

$$= \$4299.99$$

9.3 Continuous Compounding

$A = 2P$

Example 6: How long does it take for an investment to double at an annual interest rate of 4% compounded continuously?

r

$$A = Pe^{rt}$$

$$2\cancel{P} = \cancel{P}e^{.04t}$$

$$\ln(2) = \ln(e^{.04t})$$

$$\ln(2) = .04t$$

$$\frac{\ln(2)}{.04} = t \Rightarrow t \approx 17.33 \text{ years}$$

Example 7: If a bank pays 8.3% interest compounded monthly, what is the effective interest rate.

$12 = n$

Actual growth rate

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Find the value of $1 after 1 year.

$$A = 1\left(1+\frac{.083}{12}\right)^{12(1)}$$

$$= 1.08623$$

Eff interest rate $= 1.08623 - 1$

$= .08623 \Rightarrow 8.623\%$

9.4 Annuities and Sinking Funds

In the last two sections we covered simple interest and compound interest. In both cases, the amount of money deposited was a lump sum and was left to accumulate interest.

Now we will do problems were payments are made on a regular basis into an account.

When a sequence of payments of equal amount are made into an account at equal intervals of time, the account is called an **annuity**.

Annuities

An annuity is a sequence of payments made at regular time intervals. We will focus on ordinary simple level-payment annuity certains. This means:

- the payments will be made at the end of each period
  ordinary
- the frequency of payments is the same as the frequency of interest compounding
  simple
- the cash flows are of equal sizes
  level-payment
- there are a fixed number of cash flows
  certain

Future Value Annuity

Example 1: On December 1st you decide to deposit \$300 into a savings account that earns 3% APR compounded monthly at the end of every month for the next three months. How much money do you have in the account at the end of these three months?

Now	1 month	2 months	3 months
	300 ⟶	⟶	$300\left(1+\frac{.03}{12}\right)^2$
		300 ⟶	$300\left(1+\frac{.03}{12}\right)^1$
			$+\ 300$
			\$902.25

Example 2: On December 1st you decide to deposit \$300 into a savings account that earns 3% APR compounded monthly at the end of every month for the next three years. How much money do you have in the account at the end of these three years?

$$300 + 300\left(1+\frac{.03}{12}\right)^1 + 300\left(1+\frac{.03}{12}\right)^2 + \cdots + 300\left(1+\frac{.03}{12}\right)^{35}$$

Geometric Series

$$1 + x + x^2 + x^3 + x^4 + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}$$

Future Value of an Annuity

- F denotes the future value of the annuity (or loan)
- R denotes the payment size
- t denotes the number of years (the term of the annuity/loan)
- r is the nominal interest rate per year
- m is the number of conversion periods per year
- Then

$$F = R\left[\frac{(1+\frac{r}{m})^{mt}-1}{\frac{r}{m}}\right]$$

Future Value of an Annuity

Example 3: Aaron recently decided to setup a retirement fund to plan for the future. He plans to deposit \$1700 into the account at the end of every 6 months until he retires 45 years from now. The retirement fund will earn 8% APR compounded semi-annually. How much money will be in the account when Aaron retires?

R

$m=2$ r t

$$F = 1700\left[\frac{\left(1+\frac{.08}{2}\right)^{2(45)}-1}{\left(\frac{.08}{2}\right)}\right]$$

$$= \$1,407,571.67$$

Example 4: Eric needs \$5000 in three years. If the annual interest rate is 9%, how much should he deposit at the end of each month to have \$5000 at the end of three years?

$= F$

$r \rightarrow$

$m = 12$

$t = 3$

$$5000 = R\left[\frac{\left(1+\frac{.09}{12}\right)^{12(3)}-1}{\left(\frac{.09}{12}\right)}\right]$$

$$R = \$121.50$$

Example 5: Erin can save \$100 per month. How long will it take to accumulate \$10000 in an ordinary annuity earning 2.7% annual interest compounded monthly?

R

F $\quad m=12 \quad t=? \quad r=.027$

$$100\cancel{00} = 1\cancel{00}\left[\frac{\left(1+\frac{.027}{12}\right)^{12t}-1}{\left(\frac{.027}{12}\right)}\right]$$

$$\left(\frac{.027}{12}\right)100 = \frac{\left(1+\frac{.027}{12}\right)^{12t}-1}{\cancel{\left(\frac{.027}{12}\right)}}\cancel{\left(\frac{.027}{12}\right)}$$

$$.225 = \left(1+\frac{.027}{12}\right)^{12t}-1$$

Example 5 continued...

$\ln(x^p) = p\ln(x)$

$$\ln(1.225) = \ln\left(\left(1 + \frac{.027}{12}\right)^{12t}\right)$$

$$\frac{\ln(1.225)}{12\cdot\ln\left(1+\frac{.027}{12}\right)} = \frac{\cancel{12}t\cdot\cancel{\ln\left(1+\frac{.027}{12}\right)}}{\cancel{12}\cdot\cancel{\ln\left(1+\frac{.027}{12}\right)}}$$

$$\boxed{\begin{array}{l} 7.5248 = t \\ \underline{\text{years}} \end{array}}$$

Example 1: Jack borrows $2000 today. He will repay the loan by making two equal payments over the next year. The payments will be made at the end of every six months. The interest is 4.1% APR compounded semi-annually. Determine the size of Jack's payments.

Now | 6 months | 1 year

$$R\left(1+\frac{.041}{2}\right)^{-1} \longleftarrow R \text{ (6 months)}$$

$$R\left(1+\frac{.041}{2}\right)^{-2} \longleftarrow R \text{ (1 year)}$$

Want $R\left(1+\frac{.041}{2}\right)^{-1} + R\left(1+\frac{.041}{2}\right)^{-2} = 2000$

$$\Rightarrow R = \frac{2000}{\left[\left(1+\frac{.041}{2}\right)^{-1}+\left(1+\frac{.041}{2}\right)^{-2}\right]} = \$1030.85$$

A Problem with Our Current Method

Example 2: Billy takes out a home loan worth \$175,000 today. He will repay the loan by making equal payments at the end of each month for the next 30 years. The interest is 5% APR compounded monthly. Determine the size of Billy's payments.

$$R\left(1+\frac{.05}{12}\right)^{-1}+R\left(1+\frac{.05}{12}\right)^{-2}+\cdots+R\left(1+\frac{.05}{12}\right)^{-360}$$

$$=175000$$

Use a geometric Series to rewrite in a compact form.

9.5 Computing Loan Payments Formula

- P denotes the principal of a loan (how much was borrowed)
- R denotes the size of the payment
- t denotes the number of years (the term of the loan)
- r is the nominal interest rate per year
- m is the number of conversion periods
- Then

$$P = R\left[\frac{1-(1+\frac{r}{m})^{-mt}}{\frac{r}{m}}\right]$$

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- R denotes the size of the payment
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P

$$P = R\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\left(\frac{r}{m}\right)}\right]$$

$$2000 = R\left[\frac{1-\left(1+\frac{.041}{2}\right)^{-2\cdot 1}}{\left(\frac{.041}{2}\right)}\right]$$

$$2000 = R\left[1.940138959\right]$$

$$\boxed{R = \$1030.85}$$

Example 4: Billy takes out a home loan worth \$175,000 today. He will repay the loan by making equal payments at the end of each month for the next 30 years. The interest is 5% APR compounded monthly. Determine the size of Billy's payments.

P

$$175000 = R\left[\frac{1-\left(1+\frac{.05}{12}\right)^{-12(30)}}{\left(\frac{.05}{12}\right)}\right]$$

Calculator $\Rightarrow$ $R = \$939.44$

\# of payments

Total Paid for the loan $= (939.44)(360)$
$= \$338198.40$

Total Interest Paid $= 338198.40 - 175000$
$= \$163198.40$

Valuing a Loan at Different Times

Example 5: Recall that Billy's home loan was for $175,000 over 30 years compounded monthly at 5% APR. Exactly 5 years after Billy takes out this loan he wins the lottery. Billy would like to pay off his home loan at this point. How much money would this cost?

How much of the loan is left to pay off after 5 years?

What is the balance of the loan after 5 years?

Know from Ex 4, $R = \$939.44$

5 years from now there are 300 payments left over the 25 year period.

$$P = 939.44\left[\frac{1-\left(1+\frac{.05}{12}\right)^{\overbrace{-12(25)}}}{\left(\frac{.05}{12}\right)}\right]$$

# of payments left

$P = \$160\,700.65$

Left to pay at year 5.

Things to Note in Billy's Example

- The total amount paid by Billy is Refinanced

$$60(\$939.44) + \$160700.65 = \$217067.05$$

- He borrowed \$175000 so the total interest charge is Refinanced

$$\$217067.05 - \$175000 = \$42067.05$$

- Had he continued to make regular payments for the full term of his loan, his total interest expense would have been \$163198.40.

- By paying off his loan early, Billy saved

$$\$163198.40 - \$42067.05 = \$121131.35$$

Refinancing Example

Example 6: Kelsey takes out a home loan with $250000 principal. She makes payments at the end of each month for 30 years. The interest is 7.2% APR compounded monthly. Ten years into the loan, Kelsey considers refinancing her loan because interest rates have dropped to 6% APR compounded monthly. How much will Kelsey save on interest charges if she refinances?

1. Find the original payment.
2. Find the original interest charges
3. Find the balance of the loan after 10 years.
4. Find the payment on refinanced loan.
5. Find the interest on refinanced loan.

Answer = 2 − 5

① $P = R\left[\dfrac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\left(\frac{r}{m}\right)}\right]$

$$250000 = R\left[\dfrac{1-\left(1+\frac{.072}{12}\right)^{-12(30)}}{\left(\frac{.072}{12}\right)}\right]$$

$\Longrightarrow R = \$1696.97$

② $\text{Interest} = (1696.97)(12)(30) - 250000$

$= \$360909.39$

③ Balance of the loan after 10 years

$$P = 1696.97\left[\frac{1-\left(1+\frac{.072}{12}\right)^{-12(20)}}{\left(\frac{.072}{12}\right)}\right]$$

$$P = \$215529.50$$

④ New payment at 6%

$$215{,}529.50 = R\left[\frac{1-\left(1+\frac{.06}{12}\right)^{-12(20)}}{\left(\frac{.06}{12}\right)}\right]$$

$$R = \$1544.12$$

⑤ Interest on refinanced loan.

Total paid = 1696.97(10)(12)
+ 1544.12(20)(12)

574225.20

Amount Saved = 610909.20 - 574225.20

= \$ 36684

Total paid for original loan

total paid on refinanced loan.

Loan Payment and Equity

Example 7: A couple has decided to purchase a \$200000 house using a down payment of \$20000. They can pay off the balance at 12% annual interest compounded monthly over 25 years.

$P = 180000$

a. What is their monthly payment?

$$180000 = R\left[\frac{1-\left(1+\frac{.12}{12}\right)^{-12(25)}}{\frac{.12}{12}}\right]$$

$R = \$1895.80$

b. What is the total interest paid?

$$\text{Interest} = \underbrace{(1895.80)}_{\text{loan payment}}\underbrace{(12)(25)}_{\text{\# of payments}} - \underbrace{180000}_{\text{loan amount}}$$

$$= \$388741.04$$

c. What is their equity in the house after 5 years?

Equity = amount put toward purchase price.

Find the balance of the loan after 5 years and subtract from, the purchase price.

~~d. What is their equity in the house after 20 years?~~

$$P = 1895.80\left[\frac{1-\left(1+\frac{.12}{12}\right)^{-12(20)}}{\frac{.12}{12}}\right]$$

$$P = 172,175.45$$

purchase price

balance of the loan

$$\text{Equity} = 200000 - 172175.45$$

$$= \$\,27824.55$$

Multiple Annuities

Example 1: Helen plans to invest \$1000 at the end of every month for the next five years into an account which earns 4% APR compounded monthly. After these five years Helen plans to invest \$2500 at the end of every year for the next 30 years into an account which earns 3% APR compounded yearly. How much money does Helen have combined in the two accounts 35 years from now?

First 5 years

$$F = 1000\left[\frac{\left(1+\frac{.04}{12}\right)^{5(12)}-1}{\left(\frac{.04}{12}\right)}\right]$$

years 6-35

$$F = 2500\left[\frac{\left(1+\frac{.03}{1}\right)^{30(1)}-1}{\left(\frac{.03}{1}\right)}\right]$$

\$ 66298.98 (1^{st} 5 years)

\$ 118938.54 (years 6-35)

Shift the value of the 1^{st} account from year 5 to year 35.

$$66298.98\left(1+\frac{.04}{12}\right)^{30(12)}$$

$$+\ 118938.54$$

\$ 338620.08

Investing to receive regular payments

Example 2: Ruth is planning ahead to finance a return to school. To pay for school, Ruth wants to invest money at the end of every 6 months for the next five years into a savings account that earns 8% APR compounded semiannually. Ruth expects to withdraw \$5000 semiannually during the following 4 years out the account to pay for her schooling. How much should she deposit every 6 months during the first five years?

Set FV = PV payment formulas

$$\underbrace{R\left[\frac{\left(1+\frac{.08}{2}\right)^{5(2)}-1}{\frac{.08}{2}}\right]}_{\text{5 years of saving}} = \underbrace{5000\left[\frac{1-\left(1+\frac{.08}{2}\right)^{-2(4)}}{\left(\frac{.08}{2}\right)}\right]}_{\text{4 years of withdraws}}$$