HW6 #1

To show $(\min(X_i), \max(X_i))$ is not Complete suff., we just need to find one function,

$$g(\min(X_i), \max(X_i)) \text{ whose mean is zero for all } \theta$$

but $g(\min, \max)$ is not ~~zero~~ constant.

Let

$$g^*(\min(X_i), \max(X_i)) = \max(X_i) - \min(X_i) - \frac{n-1}{n+1}$$

then we can verify

$\mathbb{E}\, g^* = 0$ for all θ. yet

$\max(X_i) - \min(X_i) - \frac{n-1}{n+1}$ is not zero, because

$(\max(X_i), \min(X_i))$ have a non-degenerate joint distribution

it is not a constant, thus, $\max(X_i) - \min(X_i) - \frac{n-1}{n+1}$ is not zero

[it is zero with zero prob]. we only need to show it is not zero with prob. one.

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for that matter, any Const. C

$$C \times [\max(X_i) - \min(X_i) - \frac{n-1}{n+1}]$$ also serve as

the function to show not Complete.

HW6

$X_1 \cdots X_n \sim \text{iid } N(\theta, a\theta^2)$, $\theta > 0$ is parameter

$a > 0$ given.

① put it into exp. family form.

$$f(x_1 \cdots x_n) = \left(\frac{1}{\sqrt{2\pi a}\,\theta}\right)^n e^{-\frac{\sum (X_i-\theta)^2}{2a\theta^2}} = \left(\frac{1}{\sqrt{2\pi a}\,\theta}\right)^n e^{-\frac{\sum X_i^2 - 2\sum X_i \theta + \sum \theta^2}{2a\theta^2}}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi a}\,\theta}\right)^n}_{C(\theta)} e^{-\sum X_i^2 \cdot \frac{1}{2a\theta^2} + \sum X_i \frac{1}{a\theta}} \cdot \underbrace{e^{-\frac{n}{2a}}}_{\text{Const.}}$$

We see. $W_1 = \frac{-1}{2a\theta^2}$, $T_1 = \sum X_i^2$

$W_2 = \frac{1}{a\theta}$, $T_2 = \sum X_i$

and $\left(\frac{-1}{2a\theta^2}, \frac{1}{a\theta}\right)$ for fixed a, when $\theta > 1$

do not contain a 2-d ball.

Similar to the previous ~~pd~~ problem, We need only to find one $g(\bar{x}, s^2)$ such that $\mathbb{E}\, g(\bar{x}, s^2) = 0$ for all parameters $\in$ parameter space

i.e. $(\theta, a\theta^2)$ for $\theta > 0$

but $g(\bar{x}, s^2)$ is not a constant zero.

One such g can be

$$g(\bar{x}, s^2) = \frac{n}{a+n} \bar{x}^2 - \frac{s^2}{a},$$ easy to verify $\mathbb{E}\, g = 0$ for all $\theta > 0$

but clearly $\frac{n}{a+n} \bar{x}^2 - s^2/a$ is not constant zero,

for example we know $\bar{x}, s^2$ are indep. r.v.s. So, it have nondegenerate 2-d joint dist.

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and clearly any

$$C \cdot \left[\frac{n}{a+n} \bar{x}^2 - \frac{s^2}{a}\right]$$ will also work, for example.

$$\frac{an}{a+n} \bar{x} - s^2,$$

$$n \bar{x}^2 - \frac{a+n}{a} s^2,$$ both also works as to show not Complete suff.

HW6. [6.26 (b)]

Gamma(α, β) with α known, have density

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad \beta > 0, \quad x > 0$$

first, this ~~is~~ belongs to an exp. family: $f(x) = \underbrace{\frac{1}{\Gamma(\alpha)} x^{\alpha-1}}_{h(x)} \cdot \underbrace{\frac{1}{\beta^\alpha}}_{C(\beta)} \cdot \underbrace{e^{-[x \cdot \frac{1}{\beta}]}}_{g(T(x),\beta)}$

So, for iid obs. $X_1 X_2 \cdots X_n$,

We know $\sum_{i=1}^n X_i$ is suff for $-\frac{1}{\beta} = W(\beta)$, and thus suff for β (since $-\frac{1}{\beta} \leftrightarrow \beta$ is 1 to 1 function)

Is $\sum X_i$ minimal suff ?

We examing the ratio

$$\frac{f(\mathbf{X}|\beta)}{f(\mathbf{Y}|\beta)} = \frac{\left[\frac{1}{\Gamma(\alpha)}\right]^n \prod_{i=1}^n x_i^{\alpha-1} \left(\frac{1}{\beta^\alpha}\right)^n e^{-\sum x_i \frac{1}{\beta}}}{\left[\frac{1}{\Gamma(\alpha)}\right]^n \prod y_i^{\alpha-1} \left(\frac{1}{\beta^\alpha}\right)^n e^{-\sum y_i \frac{1}{\beta}}} = \prod \left(\frac{X_i}{Y_i}\right)^{\alpha-1} \cdot e^{-\frac{1}{\beta}(\sum X_i - \sum Y_i)}$$

$$= \prod_{i=1}^n \left(\frac{X_i}{Y_i}\right)^{\alpha-1} e^{-\frac{1}{\beta}(\sum X_i - \sum Y_i)},$$

is this a function of β ?

When is this not a function of β ?

$\Rightarrow$ If $\sum X_i = \sum Y_i$ then this do not depend on β.

$\Leftarrow$ if this do not depend on β ($e^{-\frac{1}{\beta}(\sum X_i - \sum Y_i)}$ must be const.)

$\Rightarrow$ $\sum X_i - \sum Y_i$ must be zero $\Rightarrow$ $\sum X_i = \sum Y_i$.

So, $\sum X_i$ minimal suff.