A Golden Pair of Identities in the Theory of Numbers

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We find an interesting relationship between the golden ratio, the Möbius function, the Euler totient function and the natural logarithm—central players in the theory of numbers.

Introduction Let ϕ denote the golden ratio [**3**, p. 80]

$$\phi = \frac{1+\sqrt{5}}{2},$$

a constant that has historically attracted much attention, and that obeys the identity

$$\phi = 1 + \frac{1}{\phi}.$$

Let $\mathbb{Z}_{>0}$ denote the set of positive integers.

The Euler totient function $\varphi(n)$ is defined as the number of positive integers less than $n \in \mathbb{Z}_{>0}$ that are co-prime to n [**1**, p. 233].

It is a coincidence of notation that the present theorem involves two different number theoretic quantities, ϕ and φ, commonly denoted by the Greek letter phi; hence our use of both stylistic variants of the letter.

The Möbius function $\mu(n)$ is defined on $n \in \mathbb{Z}_{>0}$ as [**1**, p. 234]

$$\mu(n) = \begin{cases} 0 \text{ if } n \text{ is non} - \text{squarefree} \\ 1 \text{ if } n \text{ is squarefree, having an even number of prime factors} \\ -1 \text{ if } n \text{ is squarefree, having an odd number of prime factors.} \end{cases}$$

These two functions are related by the well-known identity [**1**, p. 235]

$$\varphi(n) = n \sum_{d|n} \frac{\mu(d)}{d}.$$

Let $\log x$ denote the natural logarithm function of $x \in \mathbb{R}_{>0}$.

Golden identities With these notations, the following theorem highlights a connection between the golden ratio and the factorization of integers that is not obvious; and displays a sort of inverse relationship between the Möbius function and Euler totient function.

THEOREM. *We have the pair of reciprocal summation identities*

$$\phi = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log\left(1 - \frac{1}{\phi^k}\right)$$

$$\frac{1}{\phi} = -\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \log\left(1 - \frac{1}{\phi^k}\right).$$

Proof. We begin the proof of the theorem by stating two well-known identities.

LEMMA 1. *For any* $n \in \mathbb{Z}_{>0}$, *the following two identities hold* [**1**, pp. 234-235]

$$\sum_{d|n} \varphi(d) = n$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

In addition to these, we use the identity [**2**, p. 20, No. 100]

$$-\log(1 - y) = \sum_{i=1}^{\infty} \frac{y^i}{i} \quad (0 < y < 1).$$

LEMMA 2. *On the interval* $0 < x < 1$, *we have the identities*

$$(1) \quad \sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \left(-\log(1 - x^k)\right) = \frac{x}{1 - x}$$

$$(2) \quad \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \left(-\log(1 - x^k)\right) = x.$$

Proof. To prove (1), observe that the left-hand side of (1) is equal to

$$\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \sum_{i=1}^{\infty} \frac{x^{ki}}{i}.$$

For any $n \geq 1$, the coefficient of x^n in the preceding expression is

$$\frac{\sum_{k|n} \varphi(k)}{n} = 1.$$

Therefore the left side of (1) is equal to the geometric series

$$x + x^2 + x^3 + \cdots = x\,(1 + x + x^2 + \cdots) = x\,\frac{1}{1-x} = \frac{x}{1-x}.$$

To prove (2), observe that the left-hand side of (2) is equal to

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} \sum_{i=1}^{\infty} \frac{x^{ki}}{i}.$$

For any $n \geq 1$, the coefficient of x^n in the preceding expression is

$$\frac{\sum_{k|n} \mu(k)}{n} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1. \end{cases}$$

Every coefficient but the first equals zero; therefore the left side of (2) is equal to x.

Now we complete the proof of the theorem. Noting that $0 < 1/\phi < 1$, we make the substitution $x = 1/\phi$ in the two identities of Lemma 2. We use the identity $1/\phi = \phi - 1$ to simplify the right-hand side of (1), as

$$\frac{1/\phi}{1 - 1/\phi} = \frac{1}{\phi - 1} = \frac{1}{1/\phi} = \phi.$$

Other identities In addition to the expressions proved above, we note the following obvious corollaries of the theorem. The first is a summation equal to unity involving ϕ, φ and μ; the second is an infinite product equal to e; and the third gives a pair of product identities equal to e^{ϕ} and $e^{\frac{1}{\phi}}$.

COROLLARY 1. *We have the identity*

$$\sum_{k=1}^{\infty} \frac{\mu(k) - \varphi(k)}{k} \log\left(1 - \frac{1}{\phi^k}\right) = 1.$$

Proof. We subtract the second identity in the theorem from the first identity, and observe that $\phi - (1/\phi) = 1$.

COROLLARY 2. *We have the identity*

$$e = \prod_{k=1}^{\infty} \left(1 - \frac{1}{\phi^k}\right)^{\frac{\mu(k) - \varphi(k)}{k}}.$$

Proof. We apply the exponential function to both sides of Corollary 1.

COROLLARY 3. *We have the pair of identities*

$$e^{\phi} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{\phi^k}\right)^{-\frac{\varphi(k)}{k}}$$

$$e^{\frac{1}{\phi}} = \prod_{k=1}^{\infty} \left(1 - \frac{1}{\phi^k}\right)^{-\frac{\mu(k)}{k}} = \frac{1}{e}\prod_{k=1}^{\infty} \left(1 - \frac{1}{\phi^k}\right)^{-\frac{\varphi(k)}{k}}.$$

Proof. We apply the exponential function to both sides of the identities in the theorem, and observe in the second expression that

$$e^{\frac{1}{\phi}} = e^{\phi-1} = \frac{1}{e}e^{\phi}.$$

We note that product identities for $e^{\frac{x}{1-x}}$ and e^x are similarly obtained from (1) and (2) of Lemma 2, respectively.

Concluding remarks The presence of ϕ and its reciprocal in the above theorem and corollaries, fitting in naturally with basic properties of φ, μ, logarithms and geometric series, is a source of curiosity for the author. One is led by such expressions to wonder, where else might the golden ratio make an appearance in the theory of numbers?

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