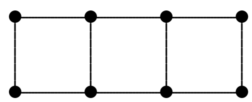
MA 514 Assignment 2. Due Wednesday September 20 at 12:50pm.

Problem 2E. A *graceful labeling* of a tree T on n vertices is a mapping $f : V(T) \to \{1, \ldots, n\}$ so that the numbers $|f(x) - f(y)|$ computed across edges $\{x, y\}$ are all different. Show that the path graphs admit graceful labelings. (It is conjectured that all trees admit graceful labelings. Can you find another class of trees with a graceful labeling?)

Problem 2F. Suppose a tree G has exactly one vertex of degree i for $2 \le i \le m$ and all other vertices have degree 1. How many vertices does G have?

Problem X2.1. Prove that every nontrivial tree has at least two vertices of degree one, by showing that the origin and terminus of a longest path in the tree both have degree one.

Problem X2.2. The *ladder graph* L_n is formed by taking two path graphs $P = v_1, v_2, \ldots, v_n$ and $P' = v_1', v_2', \ldots, v_n'$ and adding the edges (v_i, v_i') for $1 \le i \le n$. For example,

$L_4 =$ 

Show that the number $t(L_n)$ of spanning trees of the Ladder graphs satisfies the two-term recursion

$$t(L_n) = 4t(L_{n-1}) - t(L_{n-2}) \quad \text{for} \quad n \ge 3, \qquad \text{with} \quad t(L_1) = 1, t(L_2) = 4.$$

Remark. One may solve the recursion to obtain the formula

$$t(L_n) = \frac{\sqrt{3}}{6}\left((2+\sqrt{3})^n - (2-\sqrt{3})^n\right) \qquad \text{for } n \ge 1.$$

Problem X2.3.

(i) Use the Matrix Tree Theorem to prove that K_n has n^{n-2} spanning trees.
(ii) Let G be the graph obtained by deleting one edge from the complete graph K_n. Find the number of spanning trees in G.
(iii*) Optional: Let G be the graph obtained by deleting q disjoint edges from K_n. Find the number the number of spanning trees in G.