A SHORT PROOF OF STRICKLAND'S FREENESS RESULT

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1. Introduction

In [Str98], Strickland gives an algebro-geometric description of the Morava E-cohomology of a symmetric group after taking the quotient by a transfer ideal. This result plays a critical role in modern stable homotopy theory, providing an important connection between the Morava E-theories – which control height n chromatic homotopy theory – and arithmetic geometry.

An important step in producing this description is a result showing that the quotient of the E-cohomology of a symmetric group by the transfer ideal is a finitely generated free module over the coefficients. In [SS15], Schlank and the author gave an alternative proof of the algebro-geometric description using transchromatic character theory. However, the alternative proof relied on Strickland's freeness result. The purpose of this paper is to give a short proof of the freeness result.

Theorem 1.1. [Str98, Theorem 8.6] *Let E be a Morava E-theory and let I_{p^k} be the ideal in $E^0(B\Sigma_{p^k})$ that is the image of the transfer map along the inclusion $\Sigma_{p^{k-1}}^{\times p} \to \Sigma_{p^k}$. Then the quotient*

$$E^0(B\Sigma_{p^k})/I_{p^k}$$

is a finitely generated free E^0-module.

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2. Tools

In this section we describe the tools that go into the proof.

Morava E-theory and Morava K-theory. Let $E = E(\Gamma, \kappa)$ be the Morava E-theory associated to the formal group Γ over κ, a perfect field of characteristic p. Recall that E is 2-periodic with $\pi_0 E$ canonically isomorphic to the Lubin–Tate ring, which is non-canonically isomorphic to $W(\kappa)[\![u_1, \dots, u_{n-1}]\!]$, where $W(\kappa)$ is the ring of p-typical Witt vectors of κ. This is a complete local ring with maximal ideal $\mathfrak{m} = (p, u_1, \dots, u_{n-1})$. Let $K = E/\mathfrak{m}$ be the associated 2-periodic extension of the height n Morava K-theory at the prime p.

Morava E-theory of symmetric groups. Let E be a height n Morava E-theory at the prime p. [HKR00, Proposition 7.2 & Theorem 7.3] implies that $E^*(B\Sigma_m)$ is both even and finitely generated free as an E^*-module of rank equal to the number of conjugacy classes of pairwise commuting n-tuples of p-power order elements of Σ_m. [Str98, Corollary 3.8] implies that $K^0(B\Sigma_m) \cong K^0 \otimes_{E^0} E^0(B\Sigma_m)$.

The transfer ideal. Let $\sigma_{i,j}\colon \Sigma_i \times \Sigma_j \to \Sigma_{i+j}$ be the inclusion. Strickland considers the ideal $I_m \subseteq E^0(B\Sigma_m)$ given by

$$I_m = \mathrm{im}\Big(\bigoplus_{\substack{i+j=m \\ i,j>0}} E^0(B\Sigma_i \times B\Sigma_j) \xrightarrow{\oplus \mathrm{Tr}_{\sigma_{i,j}}} E^0(B\Sigma_m)\Big),$$

where $\mathrm{Tr}_{\sigma_{i,j}}$ is the transfer map. [Str98, Lemmas 8.10 & 8.11] imply that when $m \neq p^k$ for some k, then $I_m = E^0(B\Sigma_m)$ and that when $m = p^k$ then I_{p^k} is also the image of the transfer along the inclusion $\Sigma^p_{p^{k-1}} \subseteq \Sigma_{p^k}$.

Ambidexterity. The $K(n)$-local vanishing of the Tate construction [GS96] (see also [Str00]) produces a canonical isomorphism

$$K_0(BG) \xrightarrow{\mathrm{nm}_G} K^0(BG).$$

This isomorphism takes restriction maps to transfer maps: if $f\colon H \hookrightarrow G$ is an injection then the following diagram commutes

$$\begin{array}{ccc} K_0(BH) & \xrightarrow{\mathrm{nm}_H} & K^0(BH) \\ {\scriptstyle K_0(Bf)}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_f} \\ K_0(BG) & \xrightarrow{\mathrm{nm}_G} & K^0(BG). \end{array} \tag{2.1}$$

Nakayama's lemma. Assume $(R, \mathfrak{m})$ is a local ring. One version of Nakayama's lemma says that if M is a finitely generated R-module and $a_1, \dots, a_k \in M$ have the property that their images in $M/\mathfrak{m}M$ are a collection of generators, then $a_1, \dots, a_k$ are generators of M.

3. Freeness of $E^0(B\Sigma_{p^k})/I_{p^k}$

Let $K_0[\Sigma] = \bigoplus_{m\geq 0} K_0(B\Sigma_m)$. This is a commutative graded ring with product induced by $K_0(B\sigma_{i,j})$. Let $K^0[\Sigma] = \bigoplus_{m\geq 0} K^0(B\Sigma_m)$. This is a commutative graded ring with product induced by the transfer along $\sigma_{i,j}$.

Proposition 3.1. *Ambidexterity induces a canonical isomorphism of commutative graded rings*

$$K_0[\Sigma] \cong K^0[\Sigma].$$

Proof. This follows from 2.1 applied to $\sigma_{i,j}$, which gives rise to the products on each side. □

Theorem 3.2. *The E^0-module $E^0(B\Sigma_{p^k})/I_{p^k}$ is a finitely generated free E^0-module.*

Proof. In [Str98, Proposition 5.1], Strickland gives a short proof that $K_0[\Sigma]$ is a graded polynomial ring over κ. This was proven using other methods by Kashiwabara in [Kas98].

By Proposition 3.1, $K_0[\Sigma] \cong K^0[\Sigma]$, so $K^0[\Sigma]$ is also a graded polynomial ring. Since

$$K^0 \otimes_{E^0} E^0[\Sigma] \cong K^0[\Sigma],$$

Nakayama's lemma implies that $E^0[\Sigma]$ is a graded polynomial ring over E^0 with generators in the same degrees as in $K^0[\Sigma]$: we may lift the polynomial generators of $K^0[\Sigma]$ to $E^0[\Sigma]$ and the polynomial subalgebra of $E^0[\Sigma]$ generated by these must be all of $E^0[\Sigma]$.

Now note that $I_m \subseteq E^0(B\Sigma_m)$ is generated by the decomposable degree m monomials in $E^0[\Sigma]$ since the multiplication in $E^0[\Sigma]$ uses the transfer maps in the definition of I_m. Thus $E^0(B\Sigma_m)/I_m$ is the free E^0-module on the degree m polynomial generators. □

The character theory of [HKR00] can be used to see that the rank of $E^0(B\Sigma_m)/I_m$ is equal to the number of subgroups of order m in $(\mathbb{Q}_p/\mathbb{Z}_p)^n$. This is zero when m is not a power of p.

References

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