REPORT ON E-THEORY CONJECTURES SEMINAR

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In the spring of 2013, MIT ran an E-theory conjectures seminar, wherein we posed questions about E-theory and then did our best to answer them. Since there are a lot of questions and a lot of people seem to want to know things about E-theory, it seems fruitful to record some of these questions for future reference. Participants in the seminar include: Nat Stapleton (organizer), Tobi Barthel, Saul Glasman, Rune Haugseng, Aaron Mazel-Gee, Kyle Ormsby, Tomer Schlank, Vesna Stojanoska, Sebastian Thyssen, and myself, with guest appearances by many others.

I should immediately take responsibility for any nonsense that's made its way into these notes. Of course, conjectures are conjectures and if there were better reasons for them to true than wishful thinking, then they'd be theorems — but many poor transcriptions (mistakes, misrepresentations, typos, etc.) of otherwise great talks can be squarely blamed on me.

Contents

1. February 5th: Welcome and Introduction (Nat Stapleton, addenda by Eric Peterson)

Before embarking on the bulk of the seminar, Nat gave a welcome talk which described some appearances of p-divisible groups in algebraic topology and furthermore outlined how he hoped the semester would unfold. It makes sense to add two more things to the notes from that talk:

- Some words about what Morava E-theory itself is, for the benefit of relative newcomers.
- The basic definitions of some other terms which were oft-repeated later in the semester.

This section is thus meant more as a reference, and is accordingly offset from the rest of the notes.

1.1. Some constructions.

Morava E-theory enters the stage of algebraic topology via chromatic homotopy theory, the main thesis of which is that complex bordism can be thought of as a functor

$$\mathsf{Spectra} \xrightarrow{MU_*} \mathsf{QCoh}(\mathscr{M}_{\mathbf{fg}})$$

valued in quasicoherent sheaves over the moduli stack of 1-dimensional, commutative formal Lie groups. Many of the theorems in chromatic homotopy theory assert that this functor is "not too lossy" in a variety of different senses, and because of those results we can often lift structural facts about $\mathsf{QCoh}(\mathscr{M}_{\mathbf{fg}})$ to structural facts about the category $\mathsf{Spectra}$.

One useful such structural fact is that the geometric points of the p-primary component of $\mathscr{M}_{\mathbf{fg}}$ can be enumerated — they are each uniquely characterized by an integer invariant called the *height*, and every positive integer can be so realized. The height n of a formal group $\mathbb{G}$ over a field k of positive characteristic p can be defined via the rank of its p-torsion:

$$\dim_k \mathscr{O}_{\mathbb{G}[p]} = p^n.$$

We would like to produce a similar fact in algebraic topology, and there are many ways of going about this. Our first approach will be to use the Landweber exact functor theorem, which asserts that any flat map $\operatorname{Spec} R \to \mathscr{M}_{\mathbf{fg}}$ can be lifted to a map $MU \to A$ of ring spectra, where A is a 2-periodic ring spectrum whose homotopy satisfies $A_0 = R$. The value of A-homology on a space X is given by

$$A_* X = MU_* X \otimes_{MU_*} A_*.$$

However, we cannot apply this theorem to our situation directly, as the inclusion of a geometric point is rarely expected to be a flat map of stacks. We can instead replace it by its formal neighborhood in $\mathscr{M}_{\mathbf{fg}}$, which we do expect to be flat, though a priori there's no reason to expect this deformation space to be represented by a scheme. Theorems of Lubin and Tate tell us that this works out; given a formal group $\mathbb{G}$ as above, they calculate that...

- $\operatorname{Ext}^0(\mathbb{G}, \hat{\mathbb{G}}_a) = 0$, meaning the formal neighborhood of $\mathbb{G}$ can be realized as a space rather than a stack.
- $\operatorname{Ext}^2(\mathbb{G}, \hat{\mathbb{G}}_a) = 0$, meaning the point $\mathbb{G}$ is smooth, so the deformation space is given by a power series ring.
- $\dim_k \operatorname{Ext}^1(\mathbb{G}, \hat{\mathbb{G}}_a) = n-1$, meaning the power series ring is $(n-1)$-dimensional.

In all, this gives that the inclusion of the infinitesimal neighborhood of $\mathbb{G}$ into $\mathscr{M}_{\mathbf{fg}}$ can be described as

$$LT_n := \operatorname{Spf} \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \to \mathscr{M}_{\mathbf{fg}},$$

which is necessarily flat and so begets a ring spectrum $E(\mathbb{G})$ by Landweber's theorem. Generally, when $\mathbb{G}$ is a formal group of height n, this spectrum is referred to as E_n, or "the nth Morava E-theory," without explicit reference to $\mathbb{G}$. Moreover, once E_n has been realized as a ring spectrum, we can produce a ring spectrum realizing the geometric point itself by considering the quotient

$$E_n / \langle u_{n-1}, u_{n-2}, \ldots, u_1, p\rangle =: K(n),$$

called "the nth periodic Morava K-theory." This quotient realizes the inclusion of the geometric point into its formal neighborhood.

We can do slightly better with a second, more opaque construction. The submoduli $\mathscr{M}_{\mathbf{fg}}^{\geq n}$ of formal groups of height at least n gives a closed substack of $\mathscr{M}_{\mathbf{fg}}$, and its open complement $\mathscr{M}_{\mathbf{fg}}^{<n}$ satisfies the hypotheses of Landweber's theorem. This yields another cohomology theory $E(n)$, called a "Johnson–Wilson spectrum", which has the coefficient ring $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm}]$ with degrees $|v_i| = 2(p^i - 1)$. The Bousfield localization of

$E(n)$ with respect to the Morava K-theory $K(n)$ above is specified by the following formula (which is valid for any $L_{K(n)}X$, where X is an $E(n)$-local spectrum):

$$L_{K(n)}E(n) = \lim_I [M^0(I) \wedge E(n)] = \lim_I E(n)/\langle p^{I_0}, \dots, v_{n-1}^{I_{n-1}} \rangle.$$

Here I is a multi-index $I = (I_0, \dots, I_{n-1})$ of positive integers, and $M^0(I)$ is the associated generalized Moore spectrum. This spectrum $M^0(I)$ is known not to always exist, but it does exist for a cofinal subset of the possible values of I. This formula has the expected action on homotopy groups, and this can be seen to guarantee a map $L_{K(n)}E(n) \to E_n$ whose action on coefficient rings is given by $v_i \mapsto u_i \cdot u^{p^i - 1}$, where u is a periodicity element in $\pi_{-2}E_n$. This further indicates that the spectrum E_n splits multiplicatively as a wedge of suspensions of $L_{K(n)}E(n)$ — i.e., they contain identical information.

The benefit to this second approach is that the formula for $K(n)$-localization realizes topologically the profinite topology on LT_n. This is especially important in the setting of Morava E-homology — for instance, it is a theorem that a spectrum X satisfies $K(n)^*X = 0$ if and only if it also satisfies $E_n^*X = 0$. In terms of the Bousfield lattice, this means that the cohomological Bousfield classes of E_n and $K(n)$ are the same. This is as you would expect — a formal neighborhood of a smooth point hardly contains more information than the point itself. However, the homological Bousfield class of the Morava E-theory spectrum is identical to that of the Johnson–Wilson $E(n)$, which in turn is identical to the wedge $K(0) \vee \cdots \vee K(n)$, indicating that something has gone awry. This problem is fixed when we consider the K-theoretic localization: if we define "continuous Morava E-theory" by the formula

$$E_n^\vee(X) := \pi_* L_{K(n)}(E_n \wedge X) = \pi_* \lim_I (E_n / \langle p^{I_0}, \dots, u_{n-1}^{I_{n-1}} \rangle \wedge X),$$

then the acyclics of $E_n^\vee$ and $K(n)_*$ coincide. This alternative functor has a variety of nice properties, but it is *not* a homology functor — it is not guaranteed to carry infinite wedges to infinite sums.

Finally, and most opaquely, there is the obstruction theory of Goerss–Hopkins–Miller, which gives a third construction of the spectrum E_n. This is very complicated and technical, and it carries us far afield from our concerns for the rest of these notes, but there is substantial payoff. Their construction produces E_n as an E_∞ ring spectrum, meaning that Morava E-theory comes equipped with a theory of power operations and a well-behaved category of module spectra. Moreover, they show that the E_∞-automorphisms of E_n are exactly given by the Morava stabilizer group $\mathbb{S}_n$, which is the automorphism group of the underlying formal group $\mathbb{G}$. This gives a theory of fixed point spectra, some of which are known to be particularly interesting:

$$E_n^{h\mathbb{S}_n} = L_{K(n)}\mathbb{S}^0, \qquad E_1^{hC_2} = KO_2^\wedge, \qquad E_2^{hG_{24}} = L_{K(2)}\mathrm{TMF},$$

where G_{24} is a maximal subgroup of finite order (which is 24) in $\mathbb{S}_2$.

1.2. **Height modulation and p-divisible groups.** One of the basic phenomena of interest this semester is the interplay between different chromatic heights. The theory of p-divisible groups, also called Barsotti–Tate groups, is a robust framework for studying such questions, and so we introduce their language now. A p-divisible group is a sequence of finite, flat group schemes denoted $\mathbb{G}[p^j]$, satisfying the following properties:

- There is an exact sequence $0 \to \mathbb{G}[p^j] \to \mathbb{G}[p^{j+k}] \to \mathbb{G}[p^k]$ for each j and k.
- Denote the colimit of the above inclusions by $\mathbb{G}[p^\infty]$. The map $p : \mathbb{G}[p^\infty] \to \mathbb{G}[p^\infty]$ is an epimorphism of group schemes.

Two important points immediately arise: first, every R-point in a p-divisible group $\mathbb{G}[p^\infty]$ for R a finite ring over the base is p^j-torsion for some large j. This is important, and unfortunately it is not captured by the name. Second, every p-complete formal group gives rise to a p-divisible group by $\mathbb{G} \mapsto \mathbb{G}[p^\infty]$. Generally, every p-divisible group comes equipped with a short exact sequence

$$0 \to \mathbb{G}[p^\infty]_{\mathrm{for}} \to \mathbb{G}[p^\infty] \to \mathbb{G}[p^\infty]_{\mathrm{et}} \to 0,$$

where the cokernel is an étale group scheme and the kernel is the p-divisible group of a formal group.

We define the height of a p-divisible group by the rank of $\mathcal{O}_{\mathbb{G}[p]}$ as before. These p-divisible groups are exceptionally well-behaved with respect to height: by defining the pullback $f^*\mathbb{G}[p^\infty]$ of a p-divisible group as the pulled-back system $\{f^*\mathbb{G}[p^j]\}$, we see that the height of a p-divisible group is *constant* under base change. This is not true of formal groups. Moreover, the short exact sequence behaves well with respect to height:

$$\mathrm{ht}\,\mathbb{G}[p^\infty] = \mathrm{ht}\,\mathbb{G}[p^\infty]_{\mathrm{for}} + \mathrm{ht}\,\mathbb{G}[p^\infty]_{\mathrm{et}}.$$

In particular, this means that while a pullback of a formal group may drop in height, the height of the associated p-divisible group is merely shifted into the étale component.

Finally, we can produce these objects inside of algebraic topology. The p-adic circle can be described as the colimit of the groups $\mathbb{Z}/p^j$, each of which sits inside the circle as its p^j-torsion. Taking classifying spaces, this shows that the space $\mathbb{CP}^\infty = BS^1$ can be expressed p-adically as the colimit of the spaces $B\mathbb{Z}/p^j$. One can produce the following calculation using a Gysin sequence:

$$\mathbb{G}_{E_n}[p^j] := \operatorname{Spec} E^* B\mathbb{Z}/p^j \cong (\operatorname{Spf} E^* \mathbb{CP}^\infty)[p^j],$$

and hence the p-divisible group for Morava E-theory is realized in topology by this sequence of classifying spaces.

Having made this identification, we give three methods for modifying the height of Morava E-theory:

- In the simplest case, the inertia groupoid is an endofunctor of G-Spaces, which for a finite group G is given by $\Lambda_G : X \mapsto \coprod_{[g]} X^g$ ranging over conjugacy classes of elements of G. This functor is homotopically well-behaved, in the sense that it sends wedges to wedges and pushouts to pushouts, and hence any Borel-equivariant cohomology theory $E^*_G(-)$ can be precomposed with the inertia groupoid to give a new cohomology theory $E^*_G(\Lambda_G(-))$.

  In the Borel equivariant setting, we think of the classifying space $B\mathbb{Z}/p^j$ as a one-point space $*$ acted on by the group $\mathbb{Z}/p^j$. Via these spaces, the algebro-topological construction of a p-divisible group given above assigns $\mathbb{G}_{E_n}$ to Morava E-theory. We can use this to calculate the p-divisible group assigned to E-theory extended by the inertia groupoid:

$$\begin{aligned}\operatorname{Spec} E^*_{\mathbb{Z}/p^j} \Lambda_{\mathbb{Z}/p^j}(*/\!/(\mathbb{Z}/p^j)) &= \operatorname{Spec} E^*_{\mathbb{Z}/p^j}\left(\coprod_{g\in\mathbb{Z}/p^j} *^g\right) \\ &= \operatorname{Spec} E^*\left((E\mathbb{Z}/p^j \times_{\mathbb{Z}/p^j} *) \times \mathbb{Z}/p^j\right) = \mathbb{G}_{E_n}[p^j] \oplus \mathbb{Z}/p^j.\end{aligned}$$

  Hence, this construction gives a p-divisible group which is the direct sum of the original p-divisible group and the constant p-divisible group $\mathbb{Q}_p/\mathbb{Z}_p$.

  Question 1. How can we use this p-divisible group to learn new information regarding E_n? There is real potential here, Stapleton has already used this to provide an algebro-geometric description of the E-theory of certain centralizers of commuting tuples of elements in symmetric groups.

- More generally, if X is a topological stack with finite inertia groups, then we use the internal hom of topological stacks to give an alternative definition of the inertia groupoid:

$$\Lambda(X) = \hom(*/\!/\mathbb{Z}, X).$$

  These are the "stacky constant loops" on X, and the stack $*/\!/\mathbb{Z}$ representing the constant loops receives a quotient map from the stack representing all loops $\mathbb{R}/\!/\mathbb{Z} \to */\!/\mathbb{Z}$.[1] This carries an evident action of the circle group S^1, and taking the homotopy S^1-orbits yields the "twist construction":

$$E^* \operatorname{Twist}_G(X) = E^* |\hom(\mathbb{R}/\!/\mathbb{Z}, X)|_{hS^1}.$$

  In reality we use a p-adic version of this construction. Theee is also an explicit model for these objects, which one can use to calculate the p-divisible group associated to this object:

$$\mathbb{G}_{E_n^* \circ \operatorname{Twist}} = \mathbb{G}_{E_n^{BS^1}} \oplus_{\mathbb{Z}_p} \mathbb{Q}_p.$$

  Hence, this construction gives a p-divisible group with a height 1 étale part, but this time the extension is nontrivial.**I don't really understand the twist construction. It would be nice to say something more.**

- A more obvious way to study the interaction of two chromatic heights is to put them both in the same expression, and one way to do so is to study the spectrum $L_{K(t)}E_n$. The homotopy of this ring spectrum has been calculated to be

$$\pi_0 L_{K(t)} E_n = \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u_t^{-1}]^\wedge_{\langle p, u_1, \ldots, u_{t-1}\rangle}.$$

[1] It's worth pointing out that this is *not* an equivalence of topological stacks. A G-equivariant equivalence $X \to Y$ induces an equivalence of topological stacks $X/\!/G \to Y/\!/G$ only if the inverse map can also be made to be equivariant. This is not the case for $* \to \mathbb{R}$.

The formal group law associated to $L_{K(t)}E_n$ is given by pulling back the formal group law over E_n, so has height t — though the p-divisible group given by pullback has height n, and hence étale height $n-t$. Moreover, this difference can be detected in algebraic topology by the following pair of formulas:

$$(\mathbb{G}_{L_{K(t)}E_n})_{\text{for}}[p^j] = \operatorname{Spf} \pi_0 (L_{K(t)}E_n)^{\Sigma^\infty_+ B\mathbb{Z}/p^j},$$
$$\mathbb{G}_{L_{K(t)}E_n}[p^j] = \operatorname{Spf} \pi_0 L_{K(t)} \left(E_n^{\Sigma^\infty_+ B\mathbb{Z}/p^j}\right).$$

Nat's theory of transchromatic characters (and, in specific cases, the characters of Hopkins–Kuhn–Ravenel and of Chern) seeks to interrelate these constructions.

1.3. **E-theory and the algebraic geometry of spaces.** A recurring theme in the study of Morava E-theory is that its value on spaces X with extra structures is often most easily understood by considering the affine formal scheme $\operatorname{Spf} E_n^* X$. For instance, this is how we understood the p-divisible group given above. What follows is a field guide to other spaces with algebro-geometric interpretations:

Space X	Formal scheme $X_{E_n} = \operatorname{Spf} E_n^* X$
point	LT_n
$\mathbb{C}\mathrm{P}^\infty$	$\mathbb{G}_{E_n}$, a versal deformation of a formal group
$B\mathbb{Z}/m$	$\mathbb{G}_{E_n}[m]$
$K(\mathbb{Z}/m, q)$	$\mathrm{Alt}^q(\mathbb{G}_{E_n}[m])$
BA^*, A abelian	$\mathrm{Hom}(A, \mathbb{G}_{E_n})$
$BU(m)$	$\mathrm{Div}^+_m(\mathbb{G}_{E_n})$
$BU \times \mathbb{Z}$	$\mathrm{Div}(\mathbb{G}_{E_n})$
$B\Sigma_m$	$\mathrm{Sub}_m(\mathbb{G}_{E_n})$, modulo the transfer ideal
$\vdots$	$\vdots$

The essential thing to note is that all of the cohomology rings of these spaces come with extra operations, classically expressed through enormous intertwining formulas, and the act of taking the formal spectrum swallows two of the available operations: the ring sum and product. We will expand this list as the seminar progresses.

1.4. **Definition of the Picard group.** Another recurring point of our discussions is the Picard group of the $K(n)$-local stable category. Given an arbitrary symmetric monoidal (∞-)category, we can define three related objects:

(1) An object X is said to be invertible when there is some other object Y with $X \otimes Y \cong \mathbb{I}$, $\mathbb{I}$ the unit object. The collection of isomorphism classes of invertible objects forms a group Pic, called the Picard group.
(2) We can also consider the full (∞-)subcategory generated by the invertible objects.
(3) We can finally consider the further (∞-)subcategory given by restricting to only the invertible morphisms, called Pic.

The category Pic has the property that $\pi_0\mathsf{Pic}$ recovers the definition of Pic above as a group. This is recorded by fiber sequence $\Sigma GL_1\mathbb{I} \to \mathsf{Pic} \to \mathrm{Pic}$.

Most commonly, we will be interested in the Picard groups of the stable homotopy category, of the $K(n)$-local stable homotopy category, and of various categories of modules. The Picard group of the stable homotopy category is $\mathbb{Z}$, represented by the spheres — there are no further elements. The Picard group Pic_n of the $K(n)$-local stable category is much more complicated: up to a factor of $\mathbb{Z}/|v_n|$, it is a profinite-p-group. Because Morava K-theory is a monoidal functor, it sends invertible spectra to $\mathrm{Pic}(\mathsf{Lines}_{K_*})$, i.e., 1-dimensional vector spaces over K_*. A result of Hovey-Strickland shows that E-theory behaves monoidally when restricted to $K(n)$-locally invertible spectra, i.e., $E_n^\vee X$ is a 1-dimensional $(E_n)_*$-module. Together with the action of the stabilizer group $\mathbb{S}_n$, $E_n^\vee X$ is referred to as the "Morava module of X". The collection of Morava modules is denoted $\mathrm{Pic}_n^{\mathrm{alg}}$, and much of the effort put into this area of study is meant to name properties of the map

$$\mathrm{Pic}_n \xrightarrow{E_n^\vee} \mathrm{Pic}_n^{\mathrm{alg}}.$$

For instance:

- When this map injective? When is it surjective?

- What can be said about the finiteness of its source, target, or kernel?
- And so on...

The remainder of this document contains talk notes individual sessions of the seminar.

2. February 12th: Height Amplification as an Adjunction (Nat Stapleton)

Let's begin by discussing a few more questions related to the last talk and then jump into some character theory. One might try to understand what power operations are for E-theory composed with the inertia groupoid.

Charles Rezk and Yi Fei have computed the total power operation $E_2 \to E_2 B\Sigma_p \to E_2 B\Sigma_p / I_{tr}$ at the primes $p = 2$ and $p = 3$, using an explicit model for E_2 as an elliptic spectrum. Given these preexisting calculations, it would be interesting to try to compute what happens to the total power operation in the case of p-adic K-theory composed with the inertia groupoid: $K^\wedge_p \Lambda_G(-)_{hG}$ — where we can begin to come to grips with the general situation. Similarly, we could ask about the case of rational cohomology composed with the inertia groupoid $H\mathbb{Q}_*(\Lambda_G(-)_{hG})$ and how it is related to $K^\wedge_p$, or about rational cohomology composed with the 2-fold inertia groupoid, which may involve some relationship between $H\mathbb{Q}_*(\Lambda^2_G(-)_{hG})$ and E_2. It would be great to complete these calculations and try to understand the relationship between them and Rezk and Yi Fei's calculations.

There is a concrete way to compare cohomology theories of different heights. We can compare E_n and $L_{K(t)}E_n$ for $t < n$ by using character theory. However, the character maps do not quite land in $L_{K(t)}E_n$. One must extend the coefficients by an $L_{K(t)}E_n^0$-algebra called C_t. We construct a new cohomology theory

$$C_t^0(X) := C_t \otimes_{L_{K(t)}E_n^0} L_{K(t)}E_n^0(X).$$

The ring C_t is constructed to have the property that it is the initial $L_{K(t)}E_n^0$-algebra equipped with a chosen isomorphism $C_t \otimes \mathbb{G}_E[p^\infty] \cong (C_t \otimes \mathbb{G}_{L_{K(t)}E}[p^\infty]) \oplus \mathbb{Q}_p/\mathbb{Z}_p^{\times n-t}$. Already this ring stands to support interesting geometry:

Question 2 (Haynes Miller)**.** How does the stabilizer group $\mathbb{S}_n$ act on C_t?

Question 3. Does going down one height have a relationship to the determinant representation?

Idea 4. What happens at $n = 1$, $t = 0$? We can at least compute the ring then: it's $C_0 = \operatorname{colim} \mathbb{Q}_p(\zeta_{p^k})$.

With the intention of comparing these cohomology theories, Nat has constructed a map

$$E_n X_{hG} \xrightarrow{\Phi_t} C_t \otimes_{L_t} L_{K(t)}E_n \Lambda_G^{n-1}(X)_{hG} =: C_t \Lambda_G^{n-t}(X)_{hG}.$$

This map generalizes the character map of Hopkins-Kuhn-Ravenel, which essentially handles the case $t = 0$, and this in turn generalizes the (p-adic) Chern character, which handles the case of $t = 0$ and also $n = 1$. This is a good time to mention another question Haynes has asked:

Question 5 (Haynes Miller)**.** Is this in some sense the unit of an adjunction? It ought to look something like

$$\left\{ \begin{array}{c} \text{height } n \\ \text{stuff} \end{array} \right\} \underset{L_{K(t)} + \text{extension}}{\overset{-\circ \operatorname{Hom}(*/\!\!/\mathbb{Z}^{n-t}, -)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{height } t \\ \text{stuff} \end{array} \right\}$$

The real challenge is to identify what the "stuff" is, so that we can appropriately define these categories.

Idea 6. Height 0 stuff almost certainly ought to be rational spectra.

Idea 7. One could do something like the definition of elliptic spectrum, which is a category of spectra paired with elliptic curves that they model. We could as a first approximation take appropriate spectra paired with p-divisible groups that they model, along with maps between them relating to homomorphisms among the p-divisible groups.

Idea 8. We could replace "stuff" not with "spectra" but with (excisive?) functors from topological stacks to some kind of algebraic category — perhaps p-complete modules.

Idea 9. There is definitely such a thing as an extension of a cohomology theory from standard spaces to topological stacks, which works by defining the cohomology theory on an arbitrary topological stack to be its value on the geometric realization. We might look to the existing work for ideas about how to approach this. (In some cases, this is another phrasing of Borel equivariance, I think.)

Idea 10. It's possible that fussing with finite groups is holding us back, and we may want to be considering topological ∞-stacks instead (with ∞-groupoids acting). A baby step toward this understanding this would be attempting to make a construction using crossed modules. If you can do it for 1- and 2-types, then probably you can do it for ∞-groupoids if you're clever enough.

Motivated by the string of equations

$$\Lambda_G(*//G) = \coprod_{g \in G/\sim} BC(g), \qquad \Lambda_G(X//G) = EG \times_G \coprod_{g \in G} X^g, \qquad \Lambda_e(X//e) = X,$$

we might further ask the question

Question 11. Is there a nice category of spaces over π-finite spaces with fibers finite CW-complexes? In particular, can we make sense of the inertia groupoid in this category? The main obstacle to accomplishing this seems to be to define the internal hom-space appropriately.

3. February 25th: Power Operations and the Bousfield-Kuhn Functor (Nat Stapleton)

Let's talk about the relationship between power operations, the Bousfield-Kuhn functor, and the character maps. Beginning with a cohomology class $X \to E_n$, we can build a map

$$B\Sigma_k \times X \xrightarrow{\Delta} E\Sigma_k \times_{\Sigma_k} X^k \to (E\Sigma_k)_+ \wedge_{\Sigma_k} E_n^{\wedge k} \xrightarrow{\mu} E_n.$$

As X varies, this composite is called the total power operation,

$$P_k : E_n X \to E_n(B\Sigma_k \times X) \xleftarrow{\simeq} E_n B\Sigma_k \otimes E_n X.$$

As ever, we'll want to stick to the one case we understand: p-adic (2-adic, even) K-theory. Since $\Sigma_2 \simeq \mathbb{Z}/2$, and we have easy algebro-geometric descriptions of $K_2^\wedge B\mathbb{Z}/2$, we should be interested in the power operation

$$P_2 : K_2^\wedge X \to K_2^\wedge B\Sigma_2 \otimes_{\mathbb{Z}_2} K_2^\wedge X \cong K_2^\wedge B\mathbb{Z}/2 \otimes_{\mathbb{Z}_2} K_2^\wedge X.$$

Specifically, the algebro-geometric description of $K_2^\wedge B\mathbb{Z}/2$ states $K_2^\wedge B\Sigma_2 \cong K_2^\wedge B\mathbb{Z}/2 = K_2^\wedge[\![x]\!]/(2x + x^2)$ — and caveat lector: we are working with a nonstandard presentation of $K_2^\wedge B\Sigma_2$, stemming not from representation theory but from formal group theory, which is why this quotient doesn't look quite like you might expect. The isomorphism between the two presentations is given by $\mathbb{Z}_2[s]/(s^2 - 1) \to \mathbb{Z}_2[x]/(2x + x^2)$ by $s \mapsto x + 1$, which means that x corresponds to the normalized sign representation. That aside, one then computes that the action of P_2 is given by

$$P_2(x) = (\sigma^2(x) + \lambda^2(x)) + \lambda^2(x)s.$$

Our program for understand the total power operations works through the Bousfield-Kuhn functor, which is a map $\Phi_t : \mathsf{Spaces} \to \mathsf{Spectra}$ which factors $L_{K(t)}$ through Ω^∞:

$$\begin{tikzcd} \mathsf{Spectra} \arrow[rr, "L_{K(t)}"] \arrow[dr, "\Omega^\infty"'] & & \mathsf{Spectra}_{K(t)} \\ & \mathsf{Spaces.} \arrow[ur, "\Phi_t"'] & \end{tikzcd}$$

The total power operation can be thought of as a map $P_k : \Omega^\infty E_n^X \to \Omega^\infty E_n^{B\Sigma_k \times X}$, which does *not* come from a map of spectra, since it's not additive. Nonetheless, it is some map of spaces and so we are free to apply Φ_t, yielding a map $\Phi_t P_k : L_{K(t)} E_n^X \to L_{K(t)} E_n^{B\Sigma_k \times X}$. Since this is a map of spectra, it *is* additive. Setting $X = *$, Charles Rezk has shown in unpublished work that the composite

$$L_{K(t)} E_n^X \to L_{K(t)} E_n^{B\Sigma_k} \to (L_{K(t)} E_n)^{B\Sigma_k}$$

is null!

Let's try to work this through our favorite example of 2-adic K-theory, where we set $t = 0$ and aim to compute $\Phi_0 P_2 : \mathbb{Q}_2 \to \mathbb{Q}_2[s]/(2s + s^2)$. We find

$$\Phi_0 P_2(n) = \frac{-1}{2}(\sigma^2(n) - \lambda^2(n))s = \frac{-1}{2}\psi^2(n)s = \frac{1}{2}ns,$$

up to a lot of sign mistakes. This map is additive and multiplicative, as expected, but it's not a ring map since it doesn't send 1 to 1. This is actually OK, as the target splits as a product of rings, begetting projection-like maps of this type.

The next piece of this puzzle involves transfers from subgroups. For a subgroup $H \subseteq G$, we can draw a commuting square:

$$\begin{tikzcd}
E_n BH \arrow[r, "\varphi_H^{n-t}"] \arrow[d, "\mathrm{Tr}_H^G"] & \prod_{[\beta]\in[\mathbb{Z}_p^{n-t}, H]/\sim} C_t BC(\operatorname{im}\beta) \arrow[d] \\
E_n BG \arrow[r, "\varphi_G^{n-t}"] & \prod_{[\alpha]\in[\mathbb{Z}_p^{n-t}, G]/\sim} C_t BC(\operatorname{im}\alpha),
\end{tikzcd}$$

intertwining the character map with the transfer maps. This even comes with the formula

$$\varphi_G^{n-t}\operatorname{Tr}_{E_n}(x) = \sum_{[gH]\in(G/H)^{\operatorname{im}\alpha}/C(\operatorname{im}\alpha)} \operatorname{Tr}_{C_t}\Big|^{C_G(\operatorname{im}\alpha)}_{g^{-1}C_H(\operatorname{im} g\alpha g^{-1})g} \varphi_H^{n-t}[g\alpha g^{-1}](x).$$

Rather than ponder this, we immediately revert to our example, taking $e \subset \Sigma_2$ for our choice of $H \subset G$:

$$\begin{tikzcd}
K_2^\wedge \arrow[r, "\varphi_H^{n-t}"] \arrow[d, "\mathrm{Tr}_H^G"] & \mathbb{Q}_2 \arrow[d] \\
K_2^\wedge B\Sigma_2 \arrow[r, "\varphi_G^{n-t}"] & \mathbb{Q}_2[s]/s \times \mathbb{Q}_2[s]/(2+s),
\end{tikzcd}$$

and we find that the corner-to-corner composite is 1×0. Moreover, one computes

$$2^{-1}P_2 : \mathbb{Q}_2 \to \mathbb{Q}_2[s]/(2s+s^2) \cong \mathbb{Q}_2[s]/s \times \mathbb{Q}_2[s]/(2+s),$$

and the formula is $(2^{-1}P_2)(n) = (\sigma^2 n + \lambda^2 n, \sigma^2 n - \lambda^2 n)$. Projecting onto the second factor, including it back into the ring, and passing back along the isomorphism recovers the 1×0 formula above!

So, we formulate a conjecture:

Conjecture 12. *Consider the following diagram:*

$$\begin{tikzcd}
E_n \arrow[r, "P_{p^k}"'] \arrow[rr, bend left] & E_n B\Sigma_{p^k} \arrow[r] \arrow[d, "\varphi^{n-t}_{\Sigma_{p^k}}"] & (E_n B\Sigma_{p^k})/I_{tr} \arrow[d, "\varphi^{n-t}_{\Sigma_{p^k}}"] \\
& \prod C_t BC(\operatorname{im}\alpha) \arrow[r] & \prod C_t BC(\operatorname{im}\alpha)/I_{tr}^{[\alpha]} \arrow[d, "\text{project}"] \\
& & \prod_{\substack{\alpha \\ I_{tr}^{[\alpha]}=0}} C_t BC(\operatorname{im}\alpha). \arrow[ul, "\text{include}"]
\end{tikzcd}$$

We conjecture that the two composites landing in the bottom-left corner are both $\Phi_t P_{p^k}$. (Several checks reveal them to have the same formal properties.)

We closed with some discussion of what value basepoints brought to the internal hom-space construction under discussion at the beginning of the session.

4. March 5th: Spectral Tangent Space and Determinantal K-Theory (Eric Peterson)

Here in the Morava E-theory seminar, we're really interested in studying Morava E-theory and what this functor is telling us about the entire stable category. The theory of Bousfield localization informs us immediately that we aren't really learning about the stable category when we do so, but rather about the full subcategory of $K(n)$-local objects.

$$\begin{tikzcd} \mathsf{Spectra} \arrow[r, shift left, "L_K"] \arrow[rr, bend left, "K_*"] & \mathsf{Spectra}_K \arrow[l, shift left] \arrow[r, "K_*"] & \mathsf{VectorSpaces}_{K_*} \end{tikzcd}$$

For that matter (and for this reason), I'm really going to talk about Morava K-theory in this talk, though of course much of what I'll say can be lifted to E-theory without so much hassle. One thing that we know about Morava K-theory that we don't have for Morava E-theory is that it's a monoidal functor. Namely, there are monoidal structures on each of these categories (the smash product, the localized smash product, and the tensor product respectively) such that each rightward-facing functor is monoidal. So, if this is the diagram and the categories we care about, studying monoidal invariants of these categories is a sane thing to do, since the functors involve preserve this structure.

One invariant we can form is the Picard category, defined in the introduction as the full subcategory of monoidally-invertible objects, the luff subcategory of that with all invertible morphisms, or the Picard group of isomorphism classes of just the objects. This extends our diagram above like so:

$$\begin{tikzcd} \mathsf{Spectra} \arrow[r, shift left, "L_K"] \arrow[rr, bend left, "K_*"] & \mathsf{Spectra}_K \arrow[l, shift left] \arrow[r, "K_*"] & \mathsf{VectorSpaces}_{K_*} \\ \mathrm{Pic} \arrow[u] \arrow[r] & \mathrm{Pic}_n \arrow[u] \arrow[r] & \mathsf{Lines}_{K_*}. \arrow[u] \end{tikzcd}$$

The Picard category of $\mathsf{VectorSpaces}_{K_*}$ is the subcategory of 1-dimensional vector spaces Lines_{K_*}. The Picard group of the category $\mathsf{Spectra}$ is isomorphic to $\mathbb{Z}$, containing all the stable spheres $\mathbb{S}^n$, $-\infty < n < \infty$. Both of these are expected answers — what's unexpected is that the Picard group of $\mathsf{Spectra}_K$ is, up to a direct product factor of $\mathbb{Z}/|v_n|$, a profinite-p-group. This is a really curious fact, and it's essentially the only qualitative thing that's known about these groups in general. They've been completed in a few cases ($n = 1$ and $p \geq 2$, or $n = 2$ and $p \geq 3$, and otherwise not at all), but they appear to be connected to all kinds of interesting $K(n)$-local phenomena, so we're really interested in finding out more about them.

4.1. **An example.** It's reasonably easy to show that the map $\mathrm{Pic} \to \mathrm{Pic}_n$ is an inclusion, and so Pic_n is even an *infinite* profinite-p-group. In particular, this means that it must contain elements beyond just the standard stable spheres $\mathbb{S}^n$, which is exactly where the mystery above lies — what are these new elements? There's one family of elements that's easy to construct, so let's start with those to build intuition. This family arises from understanding the following family of cofiber sequences:

$$\begin{tikzcd} \cdots \arrow[r] & \mathbb{S}^{-1} \arrow[r] \arrow[d, "p^j"] & \mathbb{S}^{-1} \arrow[r] \arrow[d, "p^{j+1}"] & \cdots \arrow[r] & \mathbb{S}^{-1} \arrow[d] \\ \cdots \arrow[r, "p"] & \mathbb{S}^{-1} \arrow[r, "p"] \arrow[d] & \mathbb{S}^{-1} \arrow[r, "p"] \arrow[d] & \cdots \arrow[r] & p^{-1}\mathbb{S}^{-1} \arrow[d] \\ \cdots \arrow[r] & M^0(p^j) \arrow[r] & M^0(p^{j+1}) \arrow[r] & \cdots \arrow[r] & M^0(p^\infty). \end{tikzcd}$$

Each of these vertical sequences is the cofiber sequence defining a Moore spectra with top cell in dimension 0. They knit together by the horizontal maps shown, and a theorem about homotopy colimits says that if I take the colimit of each row individually, what I get on the far right is again a cofiber sequence. The top row is a sequence of identity maps, so its colimit is unchanged; the middle row is given by a sequence of multiplication-by-p maps, so its colimit is the spectrum S^{-1} with p inverted; and the bottom row doesn't have a good name yet, so we replace j with ∞ in the notation. Next, I want to study the K-local homotopy type of the inhabitants of this new cofiber sequence, specifically the middle term. Since K_* is of characteristic p, we know that multiplication by p is 0 on K-homology, but simultaneously we've constructed $p^{-1}\mathbb{S}^{-1}$ so that multiplication by p is an invertible map — hence, $p^{-1}\mathbb{S}^{-1}$

must be K-acyclic. In turn, this means that the going-around map $M^0(p^\infty) \to \mathbb{S}^0$ must be a K-local equivalence, and so we've identified $M^0(p^\infty)$ as an invertible spectrum — just not a very interesting one.

Here's a different way we could have proven this. A crucial theorem of Hopkins, Mahowald, and Sadofsky asserts that the right-hand square in the earlier diagram is a pullback — i.e., your favorite spectrum X is K-locally invertible exactly when its value under K_* lifts to Lines_{K_*}, or $\dim K_* X = 1$. So, if we can calculate $\dim K_* M^0(p^\infty)$, we'll be able to check that it's an invertible spectrum. This is easy enough to do: each of $K_* M^0(p^j)$ is a 2-dimensional graded K_*-vector space, with one generator in degree 0 and another in degree -1. The description of these spectra and the maps between them in terms of cofiber sequence tells us the action of the map $K_* M^0(p^j) \to K_* M^0(p^{j+1})$: it is an isomorphism on the top cell and multiplication by p — i.e., zero — on the bottom cell. Passing to the colimit, only the top cell survives and we see that $M^0(p^\infty)$ is invertible using this detection theorem.

The apparent downside to this other method is that it does not identify the homotopy type of $M^0(p^\infty)$, and we remain blissfully unaware that we've discovered the most trivial example of an invertible spectrum. On the other hand, it also suggests how to produce more examples of invertible spectra: modify the sequence by inserting maps which induce isomorphisms on K_*-homology. Each spectrum $M^0(p^j)$ is said to be "type 1", meaning that it supports a map $v_1^N : M^0(p^j) \to M^{-N|v_1|}(p^j)$ which on $K(1)$-homology induces multiplication by v_1^N for some $N \gg 0$. With an auxiliary computation, one shows that N can be taken to be $N = p^{j-1}$. With these maps in hand, select your favorite p-adic integer $a_\infty \in \mathbb{Z}_p$, with its digital expansion $a_\infty = \sum_{j=0}^\infty c_j p^j$. We associate a_∞ to the following sequence, given by interleaving the maps $v_1^{p^{j-1}}$ alongside the horizontal maps from before:

$$\cdots \to M^{-|v_1|a_{j-1}}(p^j) \to M^{-|v_1|a_{j-1}}(p^{j+1}) \xrightarrow{v_1^{c_j p^j}} M^{-|v_1|a_j}(p^{j+1}) \to \cdots \to \mathbb{S}^{-|v_1|a_\infty}.$$

Hopkins, Mahowald, and Sadofsky show that this assignment $a_\infty \mapsto \mathbb{S}^{-|v_1|a_\infty}$ determines an injection $\mathbb{Z}_p \to \mathrm{Pic}_1$, and moreover that when $p \geq 3$ the cosets of its image are represented by $\mathbb{S}^1, \ldots, \mathbb{S}^{|v_1|}$. So, for Pic_1, there is essentially nothing but the standard spheres and some kind of p-adic completion interpolating among them.

4.2. **Spectral tangent spaces.** Unfortunately, this is essentially the only general family of Picard elements we're aware of, and so the examples just about stop here — there's one other reliable family, called the determinantal spheres, which we'll come to in just a moment. In the meantime, this exploration has at least told us what to look for: if we want to find natural sources of elements of the K-local Picard group, we should look for natural sources of K_*-lines. One way I know to produce a K_*-line is to look at the tangent space at a smooth point of a 1-dimensional variety — and in fact, we sort of have examples of such things in homotopy theory. Namely, the spectrum $\mathbb{CP}^\infty$ has cohomology $K^*\mathbb{CP}^\infty = K^*[\![x]\!]$, which chromatic homotopy theory (and formal geometry) teaches us to think of in analogy to a 1-dimensional formal variety. So, we could hope for a diagram of the following shape:

$$\begin{tikzcd} \{\text{certain spectra}\} \arrow[r, dashed, "T_x"] \arrow[d, "K_*"] & \mathrm{Pic}_n \arrow[d, "K_*"] \\ \left\{ \begin{matrix} \text{1-dimensional} \\ \text{pointed varieties} \end{matrix} \right\} \arrow[r, "T_x"] & \mathsf{Lines}_{K_*}. \end{tikzcd}$$

The missing piece of this diagram is the top arrow — some kind of spectral construction of the tangent space. I've constructed such an arrow, and I'd like to describe it to you, but first let's recall the definition of cotangent space from algebraic geometry: a pointed affine K^*-variety is a K^*-algebra map $x : A \to K^*$. Its kernel is an ideal I, which receives a multiplication map $I \otimes_A I \to I$. The cokernel of this map is the definition of the cotangent space: $T_x^* A := I/I^2$. It turns out that we can basically make this construction again with spectra, but for technical reasons we have to work with coalgebras rather than with algebras. Namely, we have the following diagram for a pointed coalgebra spectrum C:

$$\begin{tikzcd} C \arrow[r] & M \arrow[r, "\Delta"] & M \square_C M \\ \mathbb{S} \arrow[u, "\eta"] & T_\eta C. \arrow[u] \end{tikzcd}$$

Let me explain. The pointing of the coalgebra spectrum is exact a map $\eta : \mathbb{S} \to C$. Then, M is the cofiber of η. What's more is that M is a C-comodule spectrum, and it carries a comultiplication map $M \to M \square_C M$. We define $T_\eta C$ to be the fiber of this last map.

This construction has dubious properties in general, but everything turns out to work just great in the one case we care about:

Theorem 13 (Peterson). *When C is a strict, pointed coalgebra spectrum such that* $\operatorname{Spf} K^* C \cong \hat{\mathbb{A}}^m$ *for some m, the following hold:*

(1) *The* Cotor *spectral sequence*

$$E_2^{*,*} = \mathrm{Cotor}^{*,*}_{K_* C}(K_* M, K_* M) \Rightarrow K_*(M \square_C M)$$

is concentrated in Cotor*-degree 0, collapses at E_2, and converges strongly without invisible extensions.*

(2) *Hence, there is an isomorphism $K_* T_\eta C \cong (T^*_\eta K^* C)^\vee$.*

(3) *When $m = 1$, we furthermore find an invertible spectrum $T_\eta C \in \mathrm{Pic}_n$.*

Idea 14 (Saul). These hypotheses can certainly be weakened from a strict coalgebra spectrum to instead some kind of coherently co-associative comultiplication for which a cobar construction still exists. On the other hand, all known examples are strict, so there's no apparent need for the extra technology.

Now that we have this machine, let me feed it some example spectra. The first and foremost example is $\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$, where one can compute

$$T_+ \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty \simeq \mathbb{S}^2.$$

This isn't super interesting though — we did all this work to discover new examples of invertible spectra, and we've wound up with just the 2-sphere again. To get something more interesting, we turn to a computation of Ravenel and Wilson, which states

$$\operatorname{Spf} K^* \underline{H\mathbb{Z}/p^\infty}_q \cong \hat{\mathbb{A}}^{\binom{n-1}{q-1}}.$$

There are two binomial coefficients which give 1: we can pick $q = 1$, which is actually the case $\Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$ all over again, or we can pick $q = n$. For this second choice, we find something interesting indeed — I'm going to jump ahead and say some words I haven't said before, but if you hold on just a moment I'll go back to the main thread of things. First, Lines_{K_*} has only one isomorphism type, so is not very good at distinguishing invertible spectra. Luckily, there is a factorization $\mathrm{Pic}_n \to \mathsf{Lines}_{K_*}$ of the form

$$\mathrm{Pic}_n \xrightarrow{\mathscr{K}} \mathsf{QCoh}_{\mathbb{S}_n}(LT_n) \xrightarrow{\text{special fiber}} \mathsf{Lines}_{K_*},$$

where $\mathsf{QCoh}_{\mathbb{S}_n}(LT_n)$ is a category of quasicoherent sheaves on Lubin-Tate space, equivariant against the action of the Morava stabilizer group $\mathbb{S}_n$ — whatever all those things are. Its main feature for us is that it has more than one isomorphism type in it. Hopf ring calculations allow us to identify the image of our new invertible spectrum in this middle category:

$$\mathscr{K}(\Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n) \cong \Omega^{n-1}_{LT_n/\mathbb{Z}_p},$$

i.e., it is "the determinantal bundle." This both identifies it as unable to be merely a standard sphere and also gives it its name: the determinantal sphere.

4.3. **Determinantal K-theory and horizons.** OK, back to the main story. This is already neat enough, as we've accomplished our main goal, but this is a conjectures seminar, so we're going to push a bit further. The first thing to notice is that the diagram defining the tangent spectrum can be extended to the right:

$$\begin{tikzcd} C \arrow[r] & M \arrow[r, "\Delta"] & M \square_C M \arrow[r] & M^{\square_C 3} \arrow[r] & \cdots \\ \mathbb{S} \arrow[u, "\eta"] & T_\eta C \arrow[u, "\beta"] & A_2 \arrow[u] & A_3 \arrow[u] & \cdots, \end{tikzcd}$$

where each of these spectra on the bottom is defined as the fiber of the map above it. Another theorem identifies their K-local homotopy type: $A_j \simeq (A_1)^{\wedge j}$. In the case of $\mathbb{C}\mathrm{P}^\infty$, we can make sense of this: the "filtration quotients" of this tower take the form $\mathbb{S}^0, \mathbb{S}^2, \mathbb{S}^4, \mathbb{S}^6, \ldots$ — that is, we're picking up the cellular decomposition of $\mathbb{C}\mathrm{P}^\infty$. On the other hand, we can also plug $\underline{H\mathbb{Z}/p^\infty}_n$ in, and we recover a similar decomposition of it into smash-powers of the determinantal sphere. This is pretty neat: the classical cell structure of $\underline{H\mathbb{Z}/p^\infty}_n$ is horrendously complicated, but if we expand our view of "cell complex" just a slight bit, it becomes as simple as that of $\mathbb{C}\mathrm{P}^\infty$.

Question 15. What other spectra can be efficiently decomposed K-locally using attaching maps along exotic elements of the K-local Picard group?

While we're in the neighborhood of $\mathbb{C}\mathrm{P}^\infty$, we may as well take a look around and see what other spaces and spectra near it we can also find near $\underline{H\mathbb{Z}/p^\infty}_n$ — after all, we've already found its cell structure laying around. Looking back up at the extended tower diagram, you'll see that I quietly labeled the map $\beta : T_\eta C \to M$. This is by analogy to the case of $C = \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty$, where β coincides with the inclusion $\mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^\infty$, i.e., the Bott element of the homotopy of $\mathbb{C}\mathrm{P}^\infty$. A theorem of Snaith says that this is enough to recover complex K-theory: there is an equivalence of ring spectra $KU \simeq \Sigma^\infty_+ \mathbb{C}\mathrm{P}^\infty[\beta^{-1}]$. Correspondingly, we can form a spectrum I'll call "determinantal K-theory":

$$R_n := \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n[\beta^{-1}].$$

This spectrum has some truly remarkable properties. Firstly, there is the following beautiful theorem of Craig Westerland:

Theorem 16 (Westerland). *There is an equivalence $R_n \simeq E_n^{hS\mathbb{S}_n}$. Here $S\mathbb{S}_n$ is the subgroup of "special" elements of the Morava stabilizer group, i.e., the kernel of the map $\det : \mathbb{S}_n \to \mathbb{Z}_p^\times$ given by considering $\mathbb{S}_n \subseteq GL_n(\mathbb{Z}_p)$ as the group of units of a certain noncommutative n-dimensional algebra. (This spectrum sometimes goes by the name of "(Mahowald's) half the K-local sphere".)*

Craig also shows a variety of properties of R_n. For instance, BU can be recovered from KU by the formula $BU = \Omega^\infty KU\langle 1\rangle$, and we can similarly define a space $W_n = \Omega^\infty R_n\langle 1\rangle$. There is a morphism $J_n : W_n \to BGL_1\mathbb{S}^0$ generalizing the complex J-homomorphism $J : BU \to BGL_1\mathbb{S}^0$, and hence a theory of determinantal Thom spectra. In particular, we can build a Thom spectrum $X_n = \mathrm{Thom}(J_n)$ in analogy to $MU = \mathrm{Thom}(J)$, and R_n carries the same universal property as KU: it is the initial X_n-oriented ring spectrum whose associated formal group (via the X_n-orientation) is multiplicative. And the list goes on from there.

This is all really fascinating, but because it's done by algebraic analogy, there's a lot missing from the picture. Most glaringly, geometry is utterly absent from the picture — here I mean the sort with vector bundles and Lie groups, not any silly formal geometry. Indeed, it turns out that analogues of the spaces $\Omega SU(m)$ (which filter $\Omega SU \simeq BU$) are easy to construct, but analogues of $BU(m)$ are not yet known.

Conjecture 17. *There is an analogous notion of "vector bundle" which is classified by analogs of the spaces $BU(m)$. The come with associated spherical bundles, which are fibered in determinantal spheres.*

Here's another place where this filtration might be coming from: the space $\mathbb{C}\mathrm{P}^\infty$ can be realized differently as the classifying space $BU(1)$. I haven't mentioned it yet, but this program also goes through for $BO(1)$ if we use $K(\infty) = H\mathbb{F}_2$ as our homology theory. The tangent space of $BU(1)$ was calculated to be $\mathbb{S}^2$, and the tangent space of $BO(1)$ can be calculated to be (2-adic) S^1 — and it's perhaps no accident that $S^2 = \Sigma U(1)$ and $S^1 = \Sigma O(1)$. Spaces of the form BG come to us as simplicial objects, and so inherit a skeletal filtration, whose filtration quotients look exactly like (shifts of) the filtration quotients we've come up with. In our situation, this lets us guess at what "$G(1)$" should be:

Conjecture 18. *There is an A_∞ multiplication on $G(1) = \Sigma^{-1} T_+ \Sigma^\infty_+ \underline{H\mathbb{Z}/p^\infty}_n$. Moreover, this suspension spectrum is realized as $BG(1)$.*

Idea 19. Actually, Craig and I seem to have mostly sorted out this positively. Specifically, $\Sigma^{-1} T_\eta C$ can also be written as $\mathrm{Cotor}_C(\mathbb{S}, \mathbb{S})$, which Koszul duality tells us carries the structure of an A_∞-algebra. There is a natural map comparing $B\,\mathrm{Cotor}_C(\mathbb{S}, \mathbb{S})$ and C itself, arising from the co/unit of an adjunction, and this can be checked to be nonvanishing on the algebraic tangent space, so a K-local isomorphism.

As a closing remark, Craig's paper on this contains several more conjectures and questions connecting it to a great many things, including certain questions in algebraic K-theory. The hungry reader should look for more things there.

5. March 12th: Orientations of Derived p-Divisible Groups (Jacob Lurie)

In this talk we're going to concern ourselves with derived algebraic geometry, so let's start by fixing a base E_∞-ring A. A derived p-divisible group over A of height n is a functor $\mathbb{G}$ from commutative E_∞-algebras over A to topological abelian groups, satisfying the following conditions:

1. $\mathbb{G}[p^k] : E_\infty\text{-}\mathsf{Algebras}_A \to \mathsf{Spaces}$ is corepresentable by a functor of rank p^{nk}, i.e., there is an E_∞-algebra $O_K \cong A^{\times(nk)}$ and a natural equivalence $\mathbb{G}[p^k](B) \cong \mathrm{Hom}_A(O_K, B)$.
2. The group is entirely p-torsion: $\mathbb{G} = \mathrm{colim}_k \mathbb{G}[p^k]$.
3. The map $p : \mathbb{G} \to \mathbb{G}$ is an fppf surjection. (It's sometimes useful to take this to be fpqc or even something else entirely. That's fine.)

The example of a p-divisible group we've been studying all along is called the *canonical p-divisible group*, written $\mathbb{G}_{\mathrm{can}}$. In symbols, it is specified by

$$\mathbb{G}_{\mathrm{can}}[p^k] = \mathrm{Hom}(A^{B\mathbb{Z}/p^k}, -).$$

One way we could generalize this example is to replace $\mathbb{Z}/p^k$ by a general finite abelian p-group H, yielding the more general formula $\mathrm{Hom}(A^{BH}, -) \simeq \mathrm{Hom}(H^*, \mathbb{G}(-))$.

An important point when studying these constructions is that BH has more maps off it than those induced by homomorphisms — specifically, basepoint nonpreserving maps. Let Cl be the category of spaces which are homotopy equivalent to BH, where H is a (varying) finite abelian p-group. Then we can compute

$$\mathrm{Hom}_{\mathsf{Cl}}(BH, BH') = \mathrm{Hom}_{Ab}(H, H') \times BH',$$

where the latter factor picks up the image of the basepoint. We make the following definition to control this phenomenon:

Definition 20. Let $\mathbb{G}$ be a p-divisible group. There is a map $\rho_0 : (\mathsf{Cl}_*)^{\mathrm{op}} \to \mathsf{CAlg}_A$ determined by the formula $\mathrm{Hom}(H^*, \mathbb{G}(B)) = \mathrm{Hom}_{\mathsf{CAlg}_A}(\rho_0(BH), B)$. A preorientation of $\mathbb{G}$ is a map $\rho : \mathsf{Cl}^{\mathrm{op}} \to \mathsf{CAlg}_A$ factoring ρ_0 as:

$$\begin{tikzcd}
(\mathsf{Cl}_*)^{\mathrm{op}} \arrow[rr, "\rho_0"] \arrow[dr] & & \mathsf{CAlg}_A \\
& \mathsf{Cl}^{\mathrm{op}}. \arrow[ur, dashed, "\rho"] &
\end{tikzcd}$$

(Equivalently, preorientations of $\mathbb{G}$ are in bijection with maps $B\mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{G}(A)$ of topological abelian groups.)

The group $\mathbb{G}_{\mathrm{can}}$ is equipped with a canonical preorientation — and in fact, if $\mathbb{G}$ is any p-divisible group over A, then preorientations of $\mathbb{G}$ biject with the space of maps $\mathrm{Hom}(\mathbb{G}_{can}, \mathbb{G})$. The goal of introducing p-divisible groups to algebraic topology is to study transchromatic phenomena, and so we make a definition to fit that into the picture:

Definition 21. Take such a preoriented p-divisible group $\mathbb{G}$ over A. We say that $\mathbb{G}$ is oriented if for every integer m the preorienting map $\mathbb{G}_{\mathrm{can}}^{K(m)} \to \mathbb{G}_{L_{K(m)}A}$ is a monomorphism with étale quotient.

There is real content in this definition. For instance, that the preorientation corresponding to the identity map $\mathbb{G}_{\mathrm{can}} \to \mathbb{G}_{\mathrm{can}}$ is an orientation is something checked in Nat's thesis. Other examples of orientations include $\hat{\mathbb{G}}_m$ for KU, *without* p-completion. There's also loads of examples of preorientations that are not orientations — the zero map $\mathbb{G}_{\mathrm{can}} \to \mathbb{G}$ is never an orientation, and we should think of the orientation condition as a sort of nondegeneracy.

Suppose that we have an oriented p-divisible group $\mathbb{G}$ over A. For a test group $T \in \mathsf{Cl}$, we get from $\mathbb{G}$ a finite flat A-algebra $\mathrm{O}_{\mathbb{G}}(T)$ of rank $|\pi_1 T|^n$. The inclusion of any point in T begets a map $\mathrm{O}_{\mathbb{G}}(T) \to A^T$ via the preorientation ρ. The case of $\mathbb{G} = \hat{G}_m$ encourages us to think of the source as $H_H^{\mathrm{genuine}}(*, A)$ and the target as $H_H^{\mathrm{Borel}}(*, A)$, where the natural map is an Atiyah-Segal map. The definitions of pre/orientation are meant to enforce the truth of the Atiyah-Segal theorem — this is another perspective on why we picked exactly the definition we did.[2]

[2] Nat: So, the genuine G-spectrum side coincides with the side with nontrivial étale part. Lurie: Yes, and this can be made precise; maybe we'll do so shortly.

We can also use $\mathbb{G}$ to produce other things that deserve to be called "Atiyah-Segal maps".

Definition 22. Still taking $\mathbb{G}$ be an oriented p-divisible group over A, let X be a space. We define $C^*_{\mathbb{G}}(X)$, the $\mathbb{G}$-twisted cochains on X, by

$$C^*_{\mathbb{G}}(X) = \lim_{\substack{T \to X \\ T \in Cl}} O_{\mathbb{G}}(T).$$

This definition mixes classical function spectra with the theory of finite groups, since we're restricting our test objects to live in Cl. There is a natural map $C^*_{\mathbb{G}}(X) \to C^*(X;A) = A^X$, which we again think of as a kind of Atiyah-Segal map. We are interested, of course, in properties of these maps, and Jacob knows about several interesting cases already.

Theorem 23. *Suppose either that X is the p-localization of a finite complex or, if p is invertible in A, that X has finite, finitely many homotopy groups (called "π-finiteness"). In these cases, the map*

$$B \wedge_A C^*_{\mathbb{G}}(X) \to C^*_{\mathbb{G}_B}(X)$$

is a homotopy equivalence of B-modules.

This is surprising and exciting! Here is another case of interest:

Theorem 24. *If A is $K(n)$-local and we work with $\mathbb{G}_{\mathrm{can}}$, then $C^*_{\mathbb{G}}(X) \to C^*(X;A)$ is an equivalence for all spaces X. (OK, this is not* super *interesting, as there is no étale part to take into account, hence it is already true.)*

Lastly, we have a splitting theorem in the case that the cohomology theory C_t is designed to embody:

Theorem 25. *Let X be a p-finite space. If there is a splitting $\mathbb{G} = \mathbb{G}' \times (\mathbb{Q}_p/\mathbb{Z}_p)^k$, then*

$$C^*_{\mathbb{G}}(X) \simeq C^*_{\mathbb{G}'}(\mathscr{L}^k X).$$

Lastly, we pressed Jacob about conjectures over lunch. He provided us with two:

Conjecture 26. ***This first one had something to do with Adams operations becoming E_∞ after rationalization, but I can't recall the exact statement.***

Conjecture 27. *There is an equivalence*

$$\mu_p(E_n) = \mathrm{Hom}_{\mathbb{S}\text{-}\mathsf{Algebras}}(\mathbb{Z}/p, \mathrm{GL}_1 E_n) \simeq K(\mathbb{Z}/p, n).$$

Equivalently, by as-yet unpublished ambidexterity work, the function spectrum functor

$$F(-, E_n) : \mathsf{Spaces}^{\leq n}_{p\text{-fin}} \to \mathsf{CAlg}^{\mathrm{op}}_{\mathsf{E_n}}$$

is fully faithful. ***This deserves a little more motivation.***

6. March 19th: The Modular Isogeny Complex (Mark Behrens)

6.1. The modular isogeny complex.

Let's jump right in.

Definition 28 (Rezk). Set $\mathbb{G} = \mathbb{CP}^\infty_E$, and let $\mathrm{Sub}_{p^{I_1},\ldots,p^{I_s}} \mathbb{G}$ be the subgroup scheme

$$\mathrm{Sub}_{p^{I_1},\ldots,p^{I_s}} \mathbb{G} = \left\{ H_1 < \cdots < H_s < \mathbb{G} \,\middle|\, |H_j / H_{j-1}| = p^{I_j} \right\},$$

where implicitly $H_0 = e$. This scheme $\mathrm{Sub}_{p^I} \mathbb{G} = \mathrm{Spf}\, S_{p^I}$ is representable, and in fact S_{p^I} are finite E^0-algebras, free as E^0-modules. Then we set

$$K^s_{p^k} = \prod_{\substack{I_1 + \cdots + I_s = k \\ I_j > 0}} S_{p^I}.$$

These come with maps $u_i : K^s_{p^k} \to K^{s+1}_{p^k}$ by omitting the ith factor, $1 \leq i \leq s$. We define a differential $\delta = \sum (-1)^i u_i$, and we claim $(K^\bullet_{p^k}, \delta)$ is a chain complex (of modules) for each choice of k.

Note that the chain

$$K^0_{p^k} \to K^1_{p^k} \to K^2_{p^k} \to \cdots \to K^k_{p^k} \to K^{k+1}_{p^k} \to \cdots$$

when $k \geq 1$ is really

$$0 \to S_{p^k} \to \prod_{k_1+k_2=k} S_{p^{k_1},p^{k_2}} \to \cdots \to S_{p,\ldots,p} \to 0 \to \cdots,$$

i.e., it is truncated. When $k = 0$, we pick this sequence to be $E_0 \to 0 \to 0 \to \cdots$ by convention.

Our goal is to uncover the homology of this complex. One way to try to answer this question is by comparing it to the Tits building, which is a pointed simplicial set $T_\bullet \in s\mathrm{Set}_*$ given by flags:

$$T_s = \left\{ 0 = V_0 \leq V_1 \leq \cdots \leq V_s \leq \mathbb{F}_p^n = V_{s+1} \right\}.$$

Note that $T_0 = *$, so this gives a natural base point for the simplicial set. We define face and degeneracy maps by omission and duplication, with the special cases $d_0 = d_s = *$ as in the bar complex. In all, $|T_\bullet|$ is called the Tits building for $GL_n\mathbb{F}_p$.

This looks rather like our complex in the case $k = n$, so maybe we can borrow some ideas from its analysis. With some work, one can manage to compute its cohomology

$$\tilde{H}^*(T_\bullet; \mathbb{F}_p \text{ or } \mathbb{Z}_p) = \begin{cases} 0 & \text{if } * \neq n, \\ \mathrm{St} & \text{if } * = n, \end{cases}$$

where St denotes the "Steinberg representation", recording the $GL_n\mathbb{F}_p$ action on $T_\bullet$.[3]

Motivated by this classical computation, Charles posed the following conjecture:

Conjecture 29 (Charles, now proven). *The cohomology of the modular isogeny complex is of the form*

$$H^* K^\bullet_{p^k} = \begin{cases} 0 & \textit{if } * \neq k, \\ \text{something interesting} & \textit{if } * = k, \end{cases}$$

where "something interesting" is specifically an object called $C[k]^\vee$ (which is 0 when $k > n$).

When he stated the general conjecture, Charles had proven this for the modular isogeny complex in the case $n = 2$, where he had as input Katz–Mazur-type geometry of elliptic curves and the comparison between isogenies and their duals. He further noticed an exact sequence

$$0 \to C[0]^\vee \to C[1]^\vee \to C[2]^\vee \to 0.$$

Charles and Mark went on to show that this conjecture was completely true:

Conjecture 30 (Behrens, Rezk). *The conjecture holds for all n, and there is always such an exact sequence*

$$0 \to C[0]^\vee \to \cdots \to C[n]^\vee \to 0.$$

6.2. **The Dyer-Lashof algebra.** Take Y to be a spectrum, and let

$$\mathbb{P}Y = \bigvee_{i \geq 0} Y^{\wedge i}_{h\Sigma_i}$$

denote its free E_∞-algebra. There is a classical result concerning the $H\mathbb{F}_p$-homology of such a spectrum: $H_*(\mathbb{P}Y; \mathbb{F}_p)$ is the free allowable algebra over the Dyer–Lashof algebra generated by H_*Y. Here, the Dyer-Lashof algebra refers to the algebra of operations acting on the $H\mathbb{F}_p$-homology of an E_∞-algebra, which is described in terms of some symbols Q^i subject to Adem-type relations. Charles showed an analogue of this for E-theory:

Theorem 31 (Rezk). *There is a monad $\mathbb{T} : \mathrm{Mod}_{E_*} \to \mathrm{Mod}_{E_*}$ defined on the entire category of E_*-modules, together with a comparison map $\eta : \mathbb{T}E_*Y \to E_*\mathbb{P}Y$. The natural transformation η is an isomorphism if E_*Y is finite and flat. There is also a splitting $\mathbb{T} = \bigoplus \mathbb{T}\langle i \rangle$ coming from the wedge decomposition of $\mathbb{P}$:*

$$E_* Y^{\wedge i}_{h\Sigma_i} = \mathbb{T}\langle i \rangle (E_* Y).$$

[3] Aaron wanted to know what dimension this representation had.

Specialize to the case $Y = S^q$ and consider $[\mathbb{T}\langle p^k\rangle(E_* S^q)]_q = E_*(B\Sigma_{p^k})^{q\bar{\rho}}$. **What do the square brackets denote?** We have two maps through which we can study this object, the transfer and the restriction:

$$\ker \mathrm{Tr} = \Gamma_q[k] \hookrightarrow E_0(B\Sigma_{p^k})^{q\bar{\rho}} \twoheadrightarrow \Delta_q[k] = \operatorname{coker} \mathrm{Res}.$$

The direct sums

$$\Gamma_q = \bigoplus_k \Gamma_q[k], \qquad \Delta_q = \bigoplus_k \Delta_q[k]$$

can both be viewed as "Dyer-Lashof" algebras. The algebra Γ_q naturally acts on $E_q A$ for A an E_∞-ring, and likewise if B is an augmented E_∞-ring $B \to \mathbb{S}$, then Δ_q acts on $(\operatorname{Ind} E_* B)_q$.

These two algebras are closely related. The composite $\Gamma_q[k] \to \Delta_q[k]$ is injective, for q odd it's even an isomorphism, and even for q even it merely has finite cockerel. One can compute them explicitly in the case $n = 1$ and $E = K_p^\wedge$:

$$\Gamma_0 = \mathbb{Z}_p[\psi^p] \qquad \Delta_0 = \mathbb{Z}_p[\theta_p].$$

For $k = 1$, the maps $\Gamma_0[1] \to E_0 B\Sigma_p \to \Delta_0[1]$ take the form $\mathbb{Z}_p\{\psi^p\} \to \mathbb{Z}_p\{(-)^p, \theta_p\} \to \mathbb{Z}_p\{\theta_p\}$, where the first map is given by $\psi^p \mapsto (-)^p + p\theta_p$ and the second acts by projecting away.[4]

In the course of studying the E-theory Dyer-Lashof algebra, Charles has recently proven the following theorem, which can be interpreted as strengthening the Behrens-Rezk result mentioned earlier:

Theorem 32 (Charles)**.** *The algebra Γ is Koszul, meaning that $B\Gamma$ is dual to the modular isogeny complex and that the cohomology of $B\Gamma$ is concentrated in particular locations.*

Charles can also generalize the second half of the Behrens-Rezk result concerning the long exact sequence by appealing to Goodwillie calculus. This arises by applying the Goodwillie tower for the identity of Spaces_* to S^1, then feeding it to E_*. This procedure can be used to build the sequence itself, and then the fact that S^1 has no periodic homotopy means that the tower is concentrated in some range, hence there's a collapse in the spectral sequence and the $C[i]^\vee$ sequence must be exact.

Mark had to run before we could press him about any conjectures (or explain this last remark).

7. April 2nd: The Topological Hochschild Homology of E-Theory (Geoffroy Horel)

Recall that Hochschild homology for a flat algebra over a commutative ring k is a functor $\mathrm{HH}_* : k\text{-}\mathsf{Alg} \to \mathrm{Ch}_*(k)$, which takes a k-algebra A to the derived tensor product $A \otimes^{\mathbb{L}}_{A\otimes A^{\mathrm{op}}} A$ (if A is not flat, one needs to derive $A \otimes A^{\mathrm{op}}$). There is a variant of this, THH, for spectra: a functor $\mathrm{THH} : \mathbb{S}\text{-}\mathsf{Alg} \to \mathsf{Spectra}$ from *strictly associative* ring spectra. There are several different constructions of THH: for instance, for cofibrant A we can employ the formula we had earlier:

$$\mathrm{THH}(A) = A \wedge^{\mathbb{L}}_{A\wedge A^{\mathrm{op}}} A.$$

where the $\wedge^{\mathbb{L}}_{A\wedge A^{\mathrm{op}}}$ symbols denotes the derived smash product in the category of $A \wedge A^{\mathrm{op}}$-modules.

We don't actually need a strict associative structure to form this spectrum, since any A_∞-algebra can be strictified. In fact, let $\mathsf{Spectra}$ be the category of symmetric spectra with the positive model structure. For any operad $\mathscr{O}$ in simplicial sets, we have a model structure on $\mathsf{Spectra}[\mathscr{O}]$, the category of algebras over $\mathscr{O}$ in spectra, and it comes with a Quillen adjunction

$$\mathsf{Spectra} \xrightarrow{\perp} \mathsf{Spectra}[\mathscr{O}].$$

A morphism $f : \mathscr{O} \to \mathscr{P}$ of operads induces another Quillen adjunction

$$\mathsf{Spectra}[\mathscr{O}] \xrightarrow{\perp} \mathsf{Spectra}[\mathscr{P}],$$

which is a Quillen equivalence if f is an equivalence of operads.

The *cyclic bar construction* is another construction of THH which is mildly more aesthetically pleasing. Given a cofibrant $\mathbb{S}$-algebra A, we define a cyclic object $C_\bullet(A)$ by $C_n(A) = A^{\wedge(n+1)}$. The degeneracies are just given by inserting the unit $\mathbb{S} \to A$, and the face maps are all given by multiplying adjacent copies of A, except that the last one is made "cyclic" by bringing the last copy of A around to the front and then multiplying there. There is an extra

[4]We further remark that Γ and Δ are also isomorphic as algebras, but noncanonically.

operator in each degree which cyclically permutes the factors but we won't use it. Then, the geometric realization of the underlying simplicial object of $C_\bullet(A)$ gives a definition of $\mathrm{THH}(A)$ (if A is cofibrant as an associative algebra):[5]

$$|C_\bullet(A)| \simeq \mathrm{THH}(A).$$

In the case that A is even a commutative $\mathbb{S}$-algebra, MacClure–Schwängzl–Vogt show $\mathrm{THH}(A) \simeq S^1 \otimes A$, where we use the simplicial tensoring of the category of commutative $\mathbb{S}$-algebras. Hence, we have some adjunctions:

$$\mathrm{map}_{\mathrm{comm}}(THH(A), B) \simeq \mathrm{map}(S^1, \mathrm{map}_{\mathrm{comm}}(A,B)) \simeq \mathrm{map}_{\mathrm{comm}}(A, LB),$$

where $LB = B^{S^1}$ denotes the "free loops" given by the simplicial cotensoring of this same category.

However, this description is not so helpful for computations — so, how does one compute THH? One way is via the Bökstedt spectral sequence:

Theorem 33 (Bökstedt)**.** *Suppose K_* is a homology theory with Künneth isomorphisms. Then there is a spectral sequence*

$$E_2 = \mathrm{HH}_*(K_*A/K_*) \Rightarrow K_*(\mathrm{THH}(A))$$

(where we consider K_*A as an associative object in K_*-modules).

Remark 34. The spectral sequence still exists if K_* doesn't have Künneth isomorphisms. For instance there is a spectral sequence computing the stable homotopy groups of $\mathrm{THH}(A)$. the E_2-page of that spectral sequence is of the form $\mathrm{Tor}^{\pi_*(A \wedge A^{\mathrm{op}})}(\pi_*A, \pi_*A)$, but in general, this is not HH of anything. Bökstedt was able to compute $\pi_* \mathrm{THH}(H\mathbb{F}_p)$ using that spectral sequence.

This is the E-theory seminar, so let's talk about the THH of Morava E-theory. There's an edge homomorphism in the Bökstedt spectral sequence, which is associated to a map $A \to \mathrm{THH}(A)$. This is really just given by remarking that $A = C_0(A)$ means that A includes as the 0-skeleton in the model $\mathrm{THH}(A) \simeq S^1 \otimes A$. This gives maps

$$\begin{tikzcd} \mathrm{HH}_*(K_*A/K_*) \arrow[r, Rightarrow] & K_* \mathrm{THH}(A) \\ K_*A \arrow[u] & K_*A \arrow[u] \end{tikzcd}$$

to the spectral sequence. Finally, here is the theorem:

Theorem 35 (Ausoni–Rognes; proof by Horel)**.** *Taking $K = K(n)$ and $E = E_n$ and working $K(n)$-locally, then $E_n \to \mathrm{THH}(E_n)$ is an $K(n)$-local equivalence.*

The proof uses the $K(n)$-Bökstedt spectral sequence (henceforth denoted "BSS"). Then the edge homomorphism to the second page $\mathrm{HH}_*(K_*E/K_*) \xleftarrow{\sim} K_*E$ is an isomorphism (meaning that it's concentrated on the vertical axis) for the following reason: in Ravenel's green book, he computes directly that $K_*E = K_*[t_1, t_2, \ldots]/(\text{relations})$, i.e., K_*E is ind-étale over K_*. However, HH_* commutes with sequential colimits, and moreover HH_* of an étale algebra is trivial.

Question 36. What is $\mathrm{THH}(E_n)$ in the larger category? For instance, $\mathrm{THH}(KU) = KU \vee \Sigma KU_{\mathbb{Q}}$.

Idea 37. It may be the case that (at least on commutative $\mathbb{S}$-algebras) $\mathrm{THH}(E_n)$ classifies deformations of H_n along with an automorphism. This guess comes from the adjunction and equivalence

$$\mathrm{map}(S^1, \mathrm{map}(E_n, B)) \simeq \mathrm{map}(\mathrm{THH}(E_n), B) \simeq \mathrm{map}(E_n, B).$$

Remark 38. Lurie has claimed that

(1) E_n doesn't have a DAG moduli-theoretic interpretation,[6] and

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[5]The cyclic structure of $C_\bullet(A)$ is what induces the S^1-action on THH see for instance the Dwyer–Hopkins–Kan paper *Homotopy Theory of Cyclic Sets* or the nLab articles for a description of the cyclic indexing category. Also, a cute fact: this category is self-opposite, so a cyclic object is the same thing as a cocyclic object.

[6]This seemed shocking to me when I first heard it, but I feel steadily more confident about what it must mean: E_n does not have an interpretation classically without taking into account its adic topology; DAG works purely with E_∞-ring spectra; and a power of p is part of ideal in the the quotients E_n/I_n^k. This is bad news, as every such E_∞-ring spectrum with p^k vanishing must be built from Eilenberg–Mac Lane spectra in a particular way, and this excludes E_n/I_n^k from ever hoping to be E_∞.

(2) there *isn't* a derived stack representing deformations of derived formal groups.

It's not clear what this means in terms of representability properties of E_n generally.

To please Nat, let's talk about things involving finite groups to round out the talk. We'll need the fact that THH is monoidal, i.e., $\mathrm{THH}(A \wedge B) \simeq \mathrm{THH}(A) \wedge \mathrm{THH}(B)$. Moreover, if G is a finite group, then one can easily compute that $\mathrm{THH}(\Sigma^\infty_+ G) \simeq \Sigma^\infty_+ LBG$ (this remains true if G is a topological group). Hence, in the $K(n)$-local category:

$$\mathrm{THH}(E_n \wedge \Sigma^\infty_+ G) \simeq \mathrm{THH}(E_n) \wedge \Sigma^\infty_+ LBG \xrightarrow{K(n)\text{-equiv.}} E_n \wedge \Sigma^\infty_+ LBG.$$

We always have a trace map $\mathbb{K}(E_n[G]) \to \mathrm{THH}(E_n[G])$ (where $\mathbb{K}$ is algebraic K-theory).

Question 39. According to the redshift conjecture $\mathbb{K}(E_n)$ has a formal group of height $n+1$. Is this map related to Nat's character map ?

Remark 40. Unfortunately, the map lands in E_n-homology instead of E_n-cohomology of $\Sigma^\infty_+ LBG$. Maybe we can make things work by using the self duality of $\Sigma^\infty_+ BG$ in the $K(n)$-local category.

8. April 9th: Brauer Groups and Resolutions of the Sphere (Drew Heard)

8.1. **The Picard group.**

Let K_n be the category of $K(n)$-local spectra, which is symmetric monoidal with product given by $X \hat{\wedge} Y = L_{K(n)}(X \wedge Y)$. Today we're interested in its associated Picard group,

$$\mathrm{Pic}_n = \left\{ [X] \in K_n \middle| \exists Y, X \hat{\wedge} Y \simeq L_{K(n)} S^0 \right\}.$$

There is a map $\mathrm{Pic}_n \to \mathrm{Pic}_n^{\mathrm{alg}}$ given by $X \mapsto E_n^\vee X$, the target of which is referred to as the category of "Morava modules," or sometimes in symbols as $\mathrm{Pic}(\mathscr{E}\mathbb{G}_n)$ or $\mathrm{Pic}_n^{\mathrm{alg}}$. There is a subgroup $\mathrm{Pic}_n^{\mathrm{alg},0} \subseteq \mathrm{Pic}_n^{\mathrm{alg}}$ of those modules concentrated in even degrees, and this is interesting because of a cohomological computation

$$\mathrm{Pic}_n^{\mathrm{alg},0} \cong H^1(\mathbb{G}_n; E_{n,0}^\times).$$

We define an exact sequence

$$0 \to \varkappa_n \to \mathrm{Pic}_n \xrightarrow{E_n^\vee} \mathrm{Pic}_n^{\mathrm{alg}},$$

and today we're most interested in studying the group $\varkappa_n$. Incidentally, here's a rather generic and I think difficult question about the obvious other half of this morphism:

Question 41. When is this map onto? When it isn't, what is its cokernel? (Note: in all cases so far it has been computed to be surjective.)

Let's explore $\varkappa_n$, which is exactly the group of spectra X with $E_n^\vee X \cong E_n^\vee \mathbb{S}^0$ as $E^\vee E$-comodules. Very few of these $\varkappa_n$ have been identified:

- $n=1$, $p=2$: Hopkins, Mahowald, and Sadofsky showed that $\varkappa_1 = \mathbb{Z}/2$.
- $n=2$, $p=2$: Mahowald claims that this takes the form $\varkappa_2 = \mathbb{Z}/2 \times \mathbb{Z}/4 \times G$, where G is an unknown factor.
- $n=2$, $p=3$: Shimomura and Kamiya showed that this takes the form of either $\mathbb{Z}/3 \times \mathbb{Z}/3$ or $\mathbb{Z}/9$, and they accomplished this with really intensive (and seemingly disorganized) computation. Goerss, Henn, Mahowald, and Rezk determined that it is actually $\varkappa_2 = \mathbb{Z}/3 \times \mathbb{Z}/3$, and they did so in a way generalizing the Hopkins-Mahowald-Sadofsky result. In the discussion below, we'll call a generator of the left-hand factor P and one of the right-hand factor Q.

We're of course interested in a systematic study, so let's begin by investigating the Hopkins-Mahowald-Sadofsky set-up. They actually proceed in multiple distinct ways — for instance, they produce enormous quantities of these invertible spectra through geometric methods, but we're going to avoid that since geometry gets tricky at higher heights. The intriguing-for-us thing they do is to study the fiber sequence

$$L_{K(1)}\mathbb{S}^0 \to KO_2^\wedge \xrightarrow{\psi^3 - 1} KO_2^\wedge,$$

lending $KO_2^\wedge$ the name "half the $K(1)$-local sphere". If we apply $K_* = K(1)_*$ to this sequence, we get an exact triangle

$$K_* \to K_* KO_2^\wedge \to K_* KO_2^\wedge.$$

The Bott perioditicty isomorphism $K_*\Sigma^4 K \simeq K_* K$ restricts to an isomorphism of continuous $\mathbb{G}_1$-modules

$$K_*\Sigma^4 KO \simeq K_* KO,$$

and so algebraically we might also study the exact sequence

$$K_* \to K_*\Sigma^4 KO_2^\wedge \to K_*\Sigma^4 KO_2^\wedge.$$

Topologically, this is realized as the fiber sequence

$$L_{K(1)} DQ \to \Sigma^4 KO_2^\wedge \to \Sigma^4 KO_2^\wedge,$$

where it's important to note the twisting of the map $\Sigma^4 KO_2^\wedge \to \Sigma^4 KO_2^\wedge$.

At $n = 2$ and $p = 3$, GHMR want to use the same idea, so they produce a resolution of the $K(2)$-local sphere:

$$L_{K(2)}\mathbb{S}^0 \to E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \wedge E_2^{hG_{24}} \to \cdots \to \Sigma^{48} E_2^{hG_{24}},$$

where $G_{24} \subseteq \mathbb{G}_2$ is a maximal finite subgroup and SD_{16} is the dihedral group of order 16. This is longer than the height 1 resolution, and so more computationally complex — and this is to be expected.[7]

Let's try to explain where P (the left-hand generator of $\varkappa_2 = \mathbb{Z}/3 \times \mathbb{Z}/3$) comes from. We can draw a diagram:

$$\begin{tikzcd}
\varkappa_2 \arrow[r, "\tau"] \arrow[d, "\varphi"] & H^5(\mathbb{G}_2, (E_2)_4) \arrow[d] \\
\varkappa_2' \arrow[r, "\tau'"] & H^5(G_{24}, (E_2)_4).
\end{tikzcd}$$

The map τ morally detects the action of the d_5 differential in the Adams-Novikov spectral sequence on the identity class (up to tracking noncanonical isomorphisms). The map φ comes from sending a Picard element X to the spectrum $X \wedge E_2^{hG_{24}}$, and the group structure in $\varkappa_2'$ is given by smashing over the algebra spectrum $E_2^{hG_{24}}$.

Theorem 42 (Goerss–Henn–Mahowald–Rezk)**.** *The bottom-right group is $\mathbb{Z}/3$ and the map τ' is an isomorphism — injectivity comes from some collapse in the associated Adams spectral sequence, surjectivity comes from the fact that there are isomorphisms $E_*(E_2^{hG_{24}}) \simeq E_*(\Sigma^{24} E_2^{hG_{24}}) \simeq E_*(\Sigma^{48} E_2^{hG_{24}})$ which cannot be realised topologically.*

Just as with $KO_2^\wedge$, we use the isomorphism $E_*^\vee E^{hG_{24}} \simeq E^\vee \Sigma^{48} E_2^{hG_{24}}$ to produce another algebraic resolution:

$$E_* \to E_* \Sigma^{48} E^{hG_{24}} \to \cdots,$$

via the 16-fold algebraic and topological periodicities of $E^{hSD_{16}}$. For the same reasons, this extends to topological resolution $\Sigma^{48} E^{hG_{24}} \to \Sigma^{56} E^{hSD_{16}} \to \cdots$, where all the Toda brackets are still zero — but this is a resolution of the exotic spectrum P instead. What this means in terms of the square diagram is that φ is surjective. Producing Q is much harder, so for clarity's sake we'll just skip it.

Instead, let's indicate how we can generalize to $n = p - 1$ — i.e., away from $p = 3$. Let G_n be a maximal finite subgroup of $\mathbb{G}_n$; then we can build an analogous square:

$$\begin{tikzcd}
\varkappa_n \arrow[r, "\tau"] \arrow[d, "\varphi"] & H^{2p-1}(\mathbb{G}_n, (E_n)_{2p-2}) \arrow[d] \\
\varkappa_n' \arrow[r] & H^{2p-1}(G_n, (E_n)_{2p-2}) = \mathbb{Z}/p.
\end{tikzcd}$$

The map τ is again related to a differential in the Adams spectral sequence — this time it's the differential on the $(2p-1)$ page. The group $\varkappa_n'$ consists of $E_n^{hG_n}$-modules, which has topological periodicity p times its algebraic periodicity.[8]

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[7] Saul: What does it mean for this to be a resolution? It means each composite is null, that every possible Toda bracket is zero. Another way to phrase this is that X_* is a resolution when each map $X_n \to X_{n+1}$ factors as $X_n \to C_n \to X_{n+1}$, where the adjacent maps $C_n \to X_{n+1} \to C_{n+1}$ form a fiber sequence — compare this with the definition of an exact sequence in an arbitrary abelian category.

[8] See Lee Nave's thesis, titled "On the Nonexistence of Smith-Toda complexes" for more information.

Remark 43. $EO_{n,*}$ is actually known — away from the 0-line, at least. This is part of Nave's write-up. It turns out they all look like $EO_{2,*}$ — this is a really remarkable and strange fact.

Hans-Werner Henn wrote a paper titled "On the finite resolution of $K(n)$-local spheres", where he works at $n = p-1$, $n > 1$. He constructs a resolution of the form

$$L_{K(n)}\mathbb{S}^0 \to E_n^{hG_n} \vee Z_0 \to \bigvee \Sigma^? E_n^{hG_n} \vee Z_1 \to \cdots \to \bigvee \Sigma^? E_n^{hG_n} \vee Z_n \to Z_{n+1} \to \cdots \to Z_m.$$

Arguing as before, using periodicity of $E_n^{hG_n}$ we can re-resolve the resolution. The complex is too complicated to think about Toda brackets, but if one can do something clever it should be possible to get other elements of the Picard group via twisted topological resolutions, and with these a surjection $\varphi : \varkappa_{p-1} \to \mathbb{Z}/p$.[9]

Question 44. Is this surjection split, as it was at $p = 3$?

8.2. **The Brauer group.** Let's take a break from the questions about the Picard group and talk about a related invariant: the Brauer group. Let $\mathsf{Spectra}_{K(n)}$ be the category of $K(n)$-local spectra. This is symmetric monodial with monoidal product $X \otimes Y := L_{K(n)}(X \wedge Y)$. Recall that in a symmetric monoidal category with product $\otimes$ there are cannonical maps $\rho : X \to DDX$, $\nu : F(X,Y) \otimes Z \to F(X, Y \otimes Z)$ and $\otimes : F(X,Y) \otimes F(Z,W) \to F(X \otimes Z, Y \otimes W)$.

If X is a multiplicative ring spectrum with multiplication $m : X \wedge X \to X$, let X° denote the opposite ring spectrum; that is the spectrum with multiplication $m' : X \wedge X \xrightarrow{T} X \wedge X \xrightarrow{m} X$, where T is the twist map.

Definition 45. Let $DA = L_{K(n)}F(X,S)$ be the functional dual of A. Then A is a $K(n)$-locally dualizable if the natural map $\nu : DA \otimes A \to F(A,A)$ is an equivalence in $\mathsf{Spectra}_{K(n)}$.

Definition 46. A $K(n)$-local spectrum X is faithful if $X \otimes Y \simeq *$ implies that $Y \simeq *$.

Equipped with these definitions we can define Azumaya algebras over $L_{K(n)}S$.

Definition 47 (Baker-Richter-Szymik). $X \in \mathsf{Spectra}_{K(n)}$ is a *weak (topological) Azumaya algebra over* $L_{K(n)}S$ if and only if the first two over the following conditions hold, whilst X is a *(topoligcal) Azumaya algebra over* $L_{K(n)}S$ if and only if all three of them hold.

(1) X is $K(n)$-locally dualizable.
(2) The natural morphism $X \otimes X^\circ \to F(X,X)$ is a $K(n)$-local equivalence.
(3) X is faithful as a $K(n)$-local spectum.

Remark 48. The Künneth formula shows that all non-trivial $X \in \mathsf{Spectra}_{K(n)}$ are in fact faithful. Also, the Baker-Richter-Szymik definition actually works more generally; it is defined for E-local R-modules, where R is a cofibrant, commutative S-algebra.

Let Az_n denote the set of (equivalence classes of) $K(n)$-local Azumaya algebras[10]. Then we can define the Brauer equivalence relation $\approx$ on Az_n.

Definition 49. If $X_1, X_2 \in \mathrm{Az}_n$, then $X_1 \approx X_2$ if and only if there are faithful, dualizable, cofibrant $K(n)$-local spectra M_1, M_2 such that

$$X_1 \otimes F(M_1, M_1) \simeq X_2 \otimes F(M_2, M_2)$$

Theorem 50. *The set Br_n is an abelian group with multiplication induced by the smash product $\otimes$.*

In a similar way to the case for commutative rings we have that Azumaya algebras of the form $F(M,M)$ are the trivial objects in the Brauer group.

The only thing known about these groups so far is the following result of Baker-Richter-Szymik:

Theorem 51. *Suppose that $p > 2$ and $n > 1$. Then the $K(n)$-local Brauer group of $L_{K(n)}S^0$ is non-trivial.*

[9] Kyle: What is the nature of the Z_i spectra? Well, we can take the cofiber $L_{K(n)}\mathbb{S}^0 \to E_n^{hN} \to K$; then E_n^{hN} has a resolution in terms of E_n^{hH}, and the target K has a finite Adams resolution. Explicitly the Z_i's are summand in a finite wedge of E_n's.

[10] This is a set!

Recall that the Picard group of a scheme $\mathrm{Pic}(X) \simeq H^1(X;\mathbb{G}_m)$, and that there is an injection

$$\mathrm{Br}(X) \hookrightarrow H^2(X,\mathbb{G}_m)_{\mathrm{torsion}}.$$

Now we can approximate the Picard group of the $K(n)$-local stable homotopy by studying $H^1(\mathbb{G}_n,(E_n)_0^\times)$ so this suggests the following:

Idea 52. Study Br_n by finding a suitable map to $H^2(\mathbb{G}_n,(E_0)^\times)$.

The natural map to study is the one that goes $X \mapsto E_*^\vee(X)$, as with the Picard group. In fact $E_*^\vee(X)$ naturally lies in the category of "Morava modules" - E_*-modules with a compatible action of $\mathbb{G}_n$. This is symmetric monoidal, and it should be possible to construct a notion of Azumaya algebra and Brauer group in this category. (It seems you may need to restrict to X concentrated in even degrees.)

Conjecture 53. *If X is a $K(n)$-local (topological) Azumaya algebra over $L_{K(n)}S$, then $E_*^\vee(X)$ is an Azumaya algebra in the category of Morava modules.*

The map should respect the Brauer equivalence, and so should give a map on Brauer groups. One can then try and prove an equivalence between the Brauer group of Morava modules and $H^2(\mathbb{G}_n,(E_n)_0^\times)$.

Idea 54. There is perhaps a more elegant way to do this. Let R be a ring spectrum. Then there is a space $\mathbf{Br}(R)$ such that $\pi_1(\mathbf{Br}(R)) \simeq \mathrm{Br}(R)$, $\pi_2(\mathbf{Br}(R)) \simeq \mathrm{Pic}(R)$ and $\pi_3(\mathbf{Br}(R)) \simeq \mathbb{G}_m(R)$ (see Szymik's *Brauer spaces for commutative rings and structured ring spectra*). Then perhaps one can see morphisms $\mathrm{Pic}_n \to H^1(\mathbb{G}_n,(E_n)_0^\times$ and $\mathrm{Br}_n \to H^2(\mathbb{G}_n,(E_n)_0^\times$ by studying the edge homomorphisms in a suitable $K(n)$-local variant of this.

9. April 16th: The E-Theory Formal Schemes of Classical K-theory Spaces (Eric Peterson)

One of the central points of Morava E-theory is its connection to algebraic geometry through its values on certain spaces. Certainly we're all aware of the importance of $E^0\mathbb{C}\mathrm{P}^\infty$, but through this semester Nat has told us a lot about $E^0B\mathbb{Z}/p^j$; Nat and Mark both have been interested in $E^0B\Sigma_{p^j}$; and in my earlier talk we considered $E^0K(\mathbb{Z}/p^j,q)$ for $q > 1$. There's another family of spaces that others have put a lot of work into studying their E-theory which haven't been named yet: the spaces in the Ω-spectra for ordinary K-theory. There are some pretty cool things known about these spaces, and that's what I want to discuss with you today.

To save on notation, throughout this talk I'll write

$$X_E := \mathrm{Spf}\, E^0X$$

for the formal scheme associated to the E-cohomology of a space X. I should mention immediately that essentially all of this is due to Matt Ando, Mike Hopkins, and Neil Strickland, except for some easy pieces, which are due to me and my coauthors Adam Hughes and JohnMark Lau. Also also, some of the conjectures below are work in progress with Matt and Neil — I'll label which to scare you off of them. Also also also, fair warning: this will be fairly rich in computation.

9.1. KU **and schemes of divisors.** Generally speaking, the complex case is simpler than the real case, and the periodic case is simpler than the connective case, so we'll begin by considering periodic complex K-theory KU. You'll recall that the whole point of having a complex oriented cohomology theory — and E-theory is no exception — is that $\mathbb{C}\mathrm{P}^\infty_E$ becomes a formal affine variety of dimension 1, i.e., $E^0\mathbb{C}\mathrm{P}^\infty$ is a power series ring in one variable. To produce a formal scheme description of $(BU \times \mathbb{Z})_E$, we'll run through the computation of its cohomology and analyze each step along the way. As the first step, there is a map $BU(1)^{\times n} \to BU(n)$, which factors as in the following triangle

$$\begin{tikzcd}
E^0BU(1)^{\otimes n} & & E^0BU(n) \arrow[ll] \arrow[dl, "\cong"] \\
& E^0[\![x_1,\ldots,x_n]\!]^{\Sigma_n}. \arrow[ul] &
\end{tikzcd}$$

A theorem of Borel identifies the marked map as an isomorphism. The bottom ring can be identified as $E^0[\![\sigma_1, \dots, \sigma_n]\!]$, where σ_i is Newton's ith symmetric function in the x_*, which are the Chern classes coming from the line bundles on the left.

Let's work to interpret this statement. The space $BU(1)^{\times n}$ classifies a rank n-bundle equipped with a splitting into a sum of n line bundles, and the map $BU(1)^{\times n} \to BU(n)$ forgets this splitting. On cohomology, this map determines the Chern classes of the rank n bundle in terms of those of the line bundles:

$$c\left(\bigoplus_{i=1}^{n} \mathscr{L}_i\right) = \prod_{i=1}^{n}(1 + t^{-1}c_1(\mathscr{L}_i)) = 1 + \sum_{i=1}^{n} t^{-i}\sigma_i(\mathscr{L}_1, \dots, \mathscr{L}_n),$$

i.e., it takes a factored polynomial and forgets the factorization. So, we should think of the ring spanned by the symmetric functions as classifying monic polynomials of degree n, and this map as forgetting a factorization.

Monic polynomials have a special place in formal geometry: suppose that we have an E^0-algebra map $E^0[\![x]\!] \to A$, where A is finite and free as an E^0-module. The Weierstrass preparation theorem guarantees that the kernel of this map can be written uniquely as the ideal generated by a monic polynomial $p(x)$ — and this correspondence is clearly bijective. On the level of schemes, such an E^0-algebra map corresponds to a free, finite closed subscheme, which is sometimes called a Cartier divisor. By definition we thus have the identification

$$BU(n)_E = \mathrm{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty.$$

The direct sum $BU(n) \times BU(m) \to BU(n+m)$ and tensor product $BU(n) \times BU(m) \to BU(nm)$ maps also have interpretations. The sum corresponds to the sum of divisors (or product of representing polynomials). The tensor product is more complicated; you already know that the tensor product map $BU(1) \times BU(1) \to BU(1)$ yields the group structure on $\mathbb{C}\mathrm{P}_E^\infty$, and the map for generic n and m is an enrichment of this. In the case that we're taking the product of two divisors written as the sum of point divisors (i.e., when considering the precomposition $BU(1)^{\times n} \to BU(n)$), the formula is given by

$$\left(\prod_{i=1}^{n}(x - a_i)\right) * \left(\prod_{j=1}^{m}(x - b_j)\right) = \prod_{i,j}(x - (a_i +_{\mathbb{C}\mathrm{P}_E^\infty} b_j)).$$

The next step in computing the cohomology of BU is to limit along the sequence

$$\cdots \to BU(n) \to BU(n+1) \to \cdots \to BU,$$

whose maps are induced by summing with the trivial line bundle (equivalently, by sending the polynomial $f(x)$ to the polynomial $x \cdot f(x)$). This indeed gives a description of the formal scheme BU_E: it is the colimit of the formal schemes $\mathrm{Div}_n^+ \mathbb{C}\mathrm{P}_E^\infty$ induced by summing with the point divisor $[0]$. A remarkable feature of this colimit construction is that it enlarges to include ineffective divisors — this comes about essentially by allowing power series inverses. We write this stabilization as

$$BU_E = \mathrm{Div}_0 \mathbb{C}\mathrm{P}_E^\infty.$$

Finally, $BU \times \mathbb{Z}$ has a $\mathbb{Z}$'s worth of copies of BU, and hence

$$(\mathbb{Z} \times BU)_E = \underline{\mathbb{Z}} \times BU_E =: \mathrm{Div}\, \mathbb{C}\mathrm{P}_E^\infty.$$

9.2. kU **and the augmentation ideal.** Having accomplished a description of the interesting spaces associated to KU, we turn now to the connective version kU. Because KU is 2-periodic, we can easily identify the spaces in its Ω-spectrum:

k	0	1	2	3	4	$\cdots$
$\underline{kU}_{2k}$	$BU \times \mathbb{Z}$	BU	$BU\langle 4\rangle = BSU$	$BU\langle 6\rangle$	$BU\langle 8\rangle$	$\cdots$.

We have descriptions of the first two, and so we'll use this information to bootstrap our way up.

What's cool — and what uses the 2-periodicity of KU — is that these higher deloopings are given simply by taking higher connective covers of BU. This gives us a tool by which we can study BSU: it belongs to the Postnikov-type fiber sequence

$$BSU \to BU \xrightarrow{\det} BU(1).$$

This begets a short exact sequence of formal group schemes — i.e., BSU_E is the fiber of the summation map $BU_E \to BU(1)_E = \mathbb{C}\mathrm{P}^\infty_E$, which sends a divisor to the sum of all the points in its zero locus. Correspondingly, we call the fiber of this map $\mathrm{SDiv}_0 \mathbb{C}\mathrm{P}^\infty_E$, the scheme of "special divisors" which vanish under the collapse map to $\mathbb{C}\mathrm{P}^\infty_E$.

Unfortunately, we cannot proceed in this way to produce a description of $BU\langle 6\rangle_E$. It belongs to a fiber sequence $BU\langle 6\rangle \to BSU \to K(\mathbb{Z},4)$, but the analogous map $BSU \to K(\mathbb{Z},4)$ does not have an obvious expression in terms of formal schemes we already know about. We might look to existing computation for clues, which will turn out to be decidedly unhelpful, but it's worth stating nonetheless. The $H\mathbb{F}_p$ cohomology of $BU\langle 2k\rangle$ was computed for $p=2$ by Stong and for all primes by Singer; at $p=2$ it is given by

$$H\mathbb{F}_2^*(BU\langle 2k\rangle) = \frac{H\mathbb{F}_2^*(BU)}{\mathbb{F}_2[\theta_{2i} \mid \sigma_2(i-1) < k-1]} \otimes \mathrm{Op}[\mathrm{Sq}^3 \iota_{2k-3}],$$

where θ_{2i} agrees with x_i modulo decomposables and Op denotes the Steenrod subalgebra closure in $H\mathbb{F}_2^* K(\mathbb{Z}, 2k-3)$. For general p, the answer looks similar, but the right-hand tensor factor is enlarged a bit.

There's no hope of pulling out a formal scheme description of $BU\langle 2k\rangle_E$ be staring at that formula, so we instead try our other favorite method: universal property. To begin, remember that the spaces $BU \times \mathbb{Z}$ and BU are given by universal properties of a certain sort: they are both "free" formal groups on $BU(1)_E = \mathbb{C}\mathrm{P}^\infty_E$, where we crucially used the map classifying the (reduced) tautological line bundle $1-\mathscr{L} : \mathbb{C}\mathrm{P}^\infty \to BU$. Thinking of this as an element φ of $kU^2BU(1)$, we can use kU's product to build elements of higher cohomological degree: $\varphi^{\times k} \in kU^{2k}BU(1)^{\times k}$.

We would like to investigate whether $BU\langle 2k\rangle_E$ is the free formal group generated by elements of the form $\varphi^{\times k}$, subject to whatever relations they enjoy. For one, it's clear that this product is invariant under rearrangement, since multiplication of divisors and tensor of bundles are both commutative. Using the shorthand $\langle a_1,\ldots,a_k\rangle = \prod_{i=1}^k([0]-[a_i])$, we further produce the following identity using the same trick as is used to study the formal group law on $\mathbb{C}\mathrm{P}^\infty_{KU}$:

$$\begin{aligned}
\langle a_1,\ldots,a_k\rangle &= ([0]-[a_1])([0]-[a_2])([0]-[a_3])\langle a_4,\ldots,a_k\rangle \\
&= ([0]-[a_1])[a_2]([0]-[a_3])\langle a_4,\ldots,a_k\rangle + \langle a_1,a_3,\ldots,a_k\rangle \\
&= ([0]-[a_1])([a_2]-[a_2+a_3])\langle a_4,\ldots,a_k\rangle + \langle a_1,a_3,\ldots,a_k\rangle \\
\langle a_2,\ldots,a_k\rangle - \langle a_1+a_2,a_3,\ldots,a_k\rangle + \langle a_1,a_3,\ldots,a_k\rangle &= \langle a_1,a_2,a_4,\ldots,a_k\rangle - \langle a_1,a_2+a_3,a_4,\ldots,a_k\rangle + \langle a_1,a_3,\ldots,a_k\rangle.
\end{aligned}$$

This looks rather horrendous, but we learn two things from it: first, canceling the two like terms on the last line, we see that the point-divisors obey a sort of cocycle identity. Second, if we look at the third line, we see an important relation: we're not just taking symmetric powers of BU_E, but rather symmetric powers *over* $(BU \times \mathbb{Z})_E$.[11] This actually turns out to be all the relations you have to notice:

Theorem 55 (Ando, Hopkins, Strickland). *There is a suitably "free" object $C_k\mathbb{C}\mathrm{P}^\infty_E = \mathrm{Sym}^k_{\mathrm{Div}\,\mathbb{C}\mathrm{P}^\infty_E} \mathrm{Div}_0 \mathbb{C}\mathrm{P}^\infty_E$. The map $BU(1)^{\times k} \to BU\langle 2k\rangle$ factors through it:*

$$\begin{tikzcd}
BU(1)^{\times k}_E \arrow[rr] \arrow[dr] & & BU\langle 2k\rangle_E \\
& C_kBU(1)_E. \arrow[ur, dashed, "\cong"] &
\end{tikzcd}$$

Moreover, the marked map is an isomorphism of group schemes for $k \leq 3$.

A lot of work goes into this theorem: you have to show that these C_k schemes even exist; that they have the desired properties; that the map is an isomorphism for much simpler cohomology theories, where it can be checked by explicit calculation (including $H\mathbb{F}_p$); and that this then implies the isomorphism for E-theory. This second-to-last step is where things fall apart for $k \geq 4$, where we see the class $\mathrm{Sq}^7\mathrm{Sq}^3\iota_5$ appear in Singer's formula. The obvious problem with this class is that it's odd-dimensional, and so the entire theory of formal schemes stops making much

[11] Saul recognized this as a weight filtration on the symmetric powers, just like we were using the weight filtration to study $BU(n)_E$ a moment ago in a different way.

sense. However, Hughes, Lau, and I have computed the representing graded ring for the Cartier dual scheme

$$DC_k\mathbb{C}\mathrm{P}^{\infty}_{H\mathbb{F}_2} = \underline{\mathrm{Hom}}_{\mathbb{F}_2}(C_k\mathbb{C}\mathrm{P}^{\infty}_{H\mathbb{F}_2}, \mathbb{G}_m),$$

leading to the following conjecture:

Conjecture 56. *The dashed map in the above triangle becomes an isomorphism after deleting not just the odd classes but their orbit under the action of the Steenrod algebra, i.e.,*

$$C_k\mathbb{C}\mathrm{P}^{\infty}_{H\mathbb{F}_2} \cong \mathrm{Spf}\left(\frac{H\mathbb{F}_2^*(BU\langle 2k\rangle)}{\mathrm{Op}[H\mathbb{F}_2^{\mathrm{odd}}(BU\langle 2k\rangle)]}\right).$$

Question 57. This draws attention to the action of the dual Steenrod algebra on $DC_k\mathbb{C}\mathrm{P}^{\infty}_{H\mathbb{F}_2}$. This is easy to compute over a certain factor and very hard over a different factor. What is the answer for the difficult factor?

Question 58. More generally, one can ask: is there a space $X(k)$ which has the property that $X(k)_E \cong C_k\mathbb{C}\mathrm{P}^{\infty}_E$? That is to say: can this deletion be performed on the level of spaces, rather than by fixing (i.e., mangling) the cohomology ring ex post facto?

Two tools came up as we were trying to address this last question.

Idea 59 (Ando). Alexander Zabrodsky (following work of Steve Wilson) has a procedure to delete odd-dimensional phenomena from H-spaces by successive fiberings, in such a way that this preserves the H-space structure. This is much smarter than trying to delete the odd-dimensional cells manually, which may destroy the H-space structure, but it's not super clear what we end up with instead.

Idea 60 (Behrens). Again in the same neighborhood as Wilson spaces, the spaces $\underline{BP\langle m\rangle}_k$ in the Ω-spectrum for truncated Brown-Peterson theory may be relevant. They, too, come with a lot of the formal tools that make the bordism tower interesting for homotopy theorists. They also agree in some sense with the tower we have in the case of $BP\langle 1\rangle$, so they could be considered natural candidates for answering this question.

9.3. kO **away from** 2. Up until this point, I've only told you things that you could find in the literature (more or less) — and this is a conjectures seminar, so I should really be making an effort to tell you about fresh and even underripe ideas. To that end, I'm going to pass to connective *real* K-theory in order to try to make good on the title of the talk and tell you some things about BString.

There is a complicated story describing what happens when 2 is not invertible, which we'll get to if there's time. The story simplifies dramatically if 2 is invertible, and it inspires the answer at 2, so we'll start there. This assumption gives us two key ingredients:

(1) Complex conjugation $\xi : kU \to kU$ is a unipotent map of order 2, which means that away from 2 we can form a pair of projection operators $P_\pm = \frac{1\pm\xi}{2}$. These split the spectrum kU into a wedge $kU^+ \vee kU^-$, on which P_+ and P_- act by the identity respectively.
(2) There is a cofiber sequence

$$\Sigma kO \xrightarrow{\nu} kO \xrightarrow{c} kU \xrightarrow{\lambda} \Sigma^2 kO,$$

where ν is the generator of $\pi_1 kO$ and c denotes complexification. The element ν is 2-torsion, so away from 2 that map is null and this cofiber sequence splits.

What's more is that these two sequences interact: $\xi c = c$ means that c factors as $c : kO \to kU^- \to kU$, and similarly $\lambda\xi = -\lambda$ means that λ factors as $kU \to kU^+ \to \Sigma^2 kO$. A couple more minutes of fussing demonstrates that these factorizations even beget equivalences $kO \simeq kU^-$ and $kU^+ \simeq \Sigma^2 kO$. Delooping everywhere yields the following table, limited by the Ando-Hopkins-Strickland hypothesis of $k \le 3$:

k	$C_k^-\mathbb{C}\mathrm{P}^\infty_E = (\underline{kO}_{2k})_E$	$C_k\mathbb{C}\mathrm{P}^\infty_E = (\underline{kU}_{2k})_E$	$C_k^+\mathbb{C}\mathrm{P}^\infty_E = (\underline{kO}_{2k+2})_E$
1	$(Sp/U)_E$	BU_E	BSp_E
2	BSp_E	BSU_E	$(\mathrm{Spin}/SU)_E$
3	$(\mathrm{Spin}/SU)_E$	$BU\langle 6\rangle_E$	$B\mathrm{String}_E$

9.4. kO **and 2-torsion phenomena.** When working at 2, we don't have either of the crucial tools we used above, so we have to approach along an entirely different route, envisioned I think entirely by Neil. For the time being, let's work with Morava K-theory — everything will turn out to be even-concentrated, so we can get an answer for completed Morava E-theory at the end if we please. I'm also going to switch quietly to homology. It turns out that homology computations are even more tightly bound to formal scheme computations, so it should still be easy to follow along.

Let's start by computing K_*BO using the Atiyah-Hirzebruch spectral sequence,

$$(H\mathbb{F}_2)_*BO \otimes K_* \Rightarrow K_*BO.$$

The $H\mathbb{F}_2$-homology of BO is known: $(H\mathbb{F}_2)_*\mathbb{R}\mathrm{P}^\infty$ is given by $\mathbb{F}_2\{\beta_0^{\mathbb{R}}, \beta_1^{\mathbb{R}}, \ldots\}$, and $(H\mathbb{F}_2)_*BO$ is the free symmetric algebra on these classes:

$$(H\mathbb{F}_2)_*BO = \mathbb{F}_2[b_1^{\mathbb{R}}, b_2^{\mathbb{R}}, b_3^{\mathbb{R}}, \ldots].$$

A theorem of Yagita states that the first differential in such an AHSS is described by the action of the nth Milnor primitive in $\mathbb{F}_2$-homology. This is calculable and results in the differential $d b_{2i}^{\mathbb{R}} = b_{2i-2^n+1}^{\mathbb{R}}$. After the action of this family of differentials, there is little left: all the odd classes are deleted, as are the classes $b_{2^n+2i}^{\mathbb{R}}$ for $i \geq 0$ — but their squares survive, saved by the Leibniz rule. What's left is concentrated in even-degrees, so the spectral sequence collapses, and there are no zerodivisors, hence no multiplicative extensions. This yields the following formula:

$$K_*BO \cong K_*[b_{2i}^{\mathbb{R}} \mid 0 < i < 2^n] \underset{K_*[(b_{2i}^{\mathbb{R}})^2 \mid 0<i<2^n]}{\otimes} K_*[(b_{2i}^{\mathbb{R}})^2 \mid 0 < i].$$

This is the description we will recast in formal schemes. The left-hand tensor factor is easiest: it's easily recognizable as the symmetric algebra on $K_*B\mathbb{Z}/2$, which identifies it as the scheme of divisors $\mathrm{Div}_0\, \mathbb{C}\mathrm{P}^\infty_K[2]$. The right-hand factor also looks like a scheme of divisors, but we have to explain the squares in it and in the corner piece. To do so, consider the addition map $BU(1)_K \times BU(1)_K \to BU(1)_K$ given by the tensor product of bundles. Since addition is commutative, this factors through the symmetric square $BU(1)_K^{\times 2} \to (BU(1)_K)^{\times 2}_{\Sigma_2} \to BU(1)_K$ — and the middle factor can be identified with $BU(2)_K$, its map to $BU(1)_K$ induced by the determinant. The desymplectification map $BSp(1) \to BU(2)$ is null-homotopic when postcomposed with the determinant map, and so lifts to a map to the scheme-theoretic fiber $BSp(1)_K \to \mathrm{fib}(BU(2)_K \to BU(1)_K)$. This turns out to be an isomorphism.

The space $Sp(1)$ has the same homotopy type as S^3, so the cohomology of $BSp(1)$ is isomorphic to a power series ring with generator in degree 4 — i.e., it is a formal curve. The symplectification map $BU(1) \to BSp(1)$ also has a role to play: on formal schemes it sends the point-divisor $[a]$ to the pair $[a, -a]$. This map, called q, is an isogeny of formal curves of degree 2. Passing to $\mathrm{Div}_0\, BU(1)_K$ and $\mathrm{Div}_0\, BSp(1)_K$, this induces push and pull maps

$$q^* : \mathrm{Div}_0\, BSp(1)_K \leftrightarrows \mathrm{Div}_0\, BU(1) : q_*,$$

which satisfy $q_* q^* = 2$, the degree of the isogeny.

This "2" is precisely where the squares in our tensor product formula are coming from. Specifically, the tensor factor $K_*[(b_{2i}^{\mathbb{R}})^2 | 0 < i]$ is a model for $\mathrm{Div}_0\, BSp(1)_K$, where the elements are so-named because that's where they're sent by the injective map

$$\mathrm{Div}_0\, BSp(1)_K \cong \mathrm{Sch}\, K_*[(b_{2i}^{\mathbb{R}})^2 \mid 0 < i] \xrightarrow{q^*} \mathrm{Sch}\, K_*[b_{2i}^{\mathbb{R}} \mid 0 < i] \cong \mathrm{Div}_0\, BU(1)_K.$$

The leftward map, q_*, sends each generator to its square (plus higher order products). Finally, even though $BSp(1)_K$ does not carry the structure of a formal group, we can use the isogeny q to define a self-map $2 : BSp(1)_K \to BSp(1)_K$, and the corner piece we're tensoring over is modeled by $\mathrm{Div}_0\, BSp(1)_K[2]$. In all, this lets us draw the following bi-Cartesian square:

$$\begin{tikzcd}
\mathrm{Div}_0\, BSp(1)_K[2] \arrow[r, "q^*"] \arrow[d] & \mathrm{Div}_0\, BU(1)_K[2] \arrow[d] \\
\mathrm{Div}_0\, BSp(1)_K \arrow[r, "q^*"] & BO_K.
\end{tikzcd}$$

This seems cumbersome at first, but it makes the next two steps quick and easy. The point of giving a presentation by a bi-Cartesian square is that kernels of maps from the bottom corner automatically inherit such presentations as well. In analogy to our analysis of $BSU \to BU \to BU(1)$, we have a fiber sequence

$$BSO \to BO \xrightarrow{\det} BO(1),$$

which begets a cube whose top and bottom faces are both bi-Cartesian:

$$\begin{tikzcd}
\mathrm{Div}_0 BSp(1)_K[2] \arrow[rr] \arrow[dr] \arrow[dd] & & \mathrm{SDiv}_0 BU(1)_K[2] \arrow[dr] \arrow[dd] & & \\
& \mathrm{Div}_0 BSp(1)_K \arrow[rr] \arrow[dd] & & BSO_K \arrow[dd] & \\
\mathrm{Div}_0 BSp(1)_K[2] \arrow[rr] \arrow[dr] & & \mathrm{Div}_0 BU(1)_K[2] \arrow[dr] & & \\
& \mathrm{Div}_0 BSp(1)_K \arrow[rr] & & BO_K \arrow[r, "\det"] & BO(1)_K \cong BU(1)_K[2].
\end{tikzcd}$$

Here, the composite $\mathrm{Div}_0 BU(1)_K[2] \to BU(1)_K[2]$ is the summation map, and $\mathrm{Div}_0 BSp(1)_K \to BU(1)_K[2]$ is zero. We can produce a description of $B\mathrm{Spin}_K$ using the same method, but we run into a small obstacle: we have to analyze the action of the maps

$$\mathrm{SDiv}_0 BSp(1)_K \to BSO_K \to (\underline{H\mathbb{Z}/2}_2)_K \cong (BU(1)_K[2])^{\wedge 2},$$
$$\mathrm{SDiv}_0 BU(1)_K[2] \to BSO_K \to (\underline{H\mathbb{Z}/2}_2)_K \cong (BU(1)_K[2])^{\wedge 2}.$$

I don't quite know how to compute these maps, but Neil has a guess as to what they are, and I think we'll end up getting it before long:

Conjecture 61 (Neil Strickland, nearly proven)**.** *The first composite is zero. The second composite acts by $\langle a, b\rangle \mapsto a \wedge b$, where $\mathrm{SDiv}_0 BSp(1)_K$ is identified with $C_2 BSp(1)_K$.*

We now want to finish the story and compute $B\mathrm{String}_K$. This space fibers over $B\mathrm{Spin}$ with fiber $K(\mathbb{Z}, 3)$, and this sequence comes with a comparison map to another fiber sequence we already understand well:

$$\begin{tikzcd}
K(\mathbb{Z},3) \arrow[r] \arrow[d, equal] & B\mathrm{String} \arrow[r] \arrow[d] & B\mathrm{Spin} \arrow[r] \arrow[d] & K(\mathbb{Z}/2,3)_K \\
K(\mathbb{Z},3) \arrow[r] & BU\langle 6\rangle \arrow[r] & BSU. &
\end{tikzcd}$$

Kitchloo, Laures, and Wilson show that the top sequence of group schemes is short exact, and Ando, Hopkins, and Strickland show the same of the bottom sequence. There's also a map upward of fiber sequences, coming from λ rather than from c. The idea from here is to use the presentation of $B\mathrm{Spin}_K$ together with facts about how these maps λ and so on interact to deduce that $B\mathrm{String}$ has the same description as it did away from 2:

Conjecture 62 (Matt Ando, Neil Strickland)**.** *There is an isomorphism*

$$B\mathrm{String}_K = C_3^+ \mathbb{CP}^\infty_K$$

at 2 as well as away from 2.[12]

Idea 63. The Cartier duals of these schemes are supposed to tie into the study of "Σ-structures," just like the complex connective story connects to "Θ-structures". It would be nice both to establish this connection and also to understand what value Σ-structures hold for arithmetic geometers. (It's worth pointing out that Gerd Laures and Nitu Kitchloo have a paper about comparing "real structures" with $B\mathrm{String}_{K(1)}$ and $B\mathrm{String}_{K(2)}$. I don't even know

[12]Again, a warning: since there's a program in place for proving this with a large unpublished body of work sitting beneath it and at least two adults working on it, it's not appropriate to encourage others to begin thinking about it from scratch. On the other hand, lots of the ideas and questions around this project are wide open and equally interesting.

if these are the same thing, but I suspect they are. Gerd also just posted a preprint concerning characteristic classes for $\mathrm{TMF}_1(3)$, which addresses ideas in this same neighborhood.)

10. April 26th: Future Directions in Algebraic Topology (Tyler Lawson at Talbot)

Since this is a talk for a private audience, I hope you'll give me the leeway to speak a little loosely. I guess the place to start is to say that we in algebraic topology study things which we have no hope of completely understanding, and so it's certainly not reasonable to make conjectures on how we could go about doing that. On the other hand, we have a history of developing tools that find effective use in other subjects — cohomology and category theory, to name the most obvious two. Maybe this is a more reasonable subject to spend time talking about: how we might influence other fields near chromatic homotopy theory.

10.1. Equivariant homotopy theory.

Associated to a finite group G, there is a category of G-spectra. For each subgroup $H \leq G$ and G-spectrum X, there is an associated fixed point spectrum X^H, and these fit together in complicated ways determined by the subgroup lattice of G — collectively, they look rather like a Mackey functor. In principle, if we wanted to compute something about these objects, we could make use of the Tom Dieck splitting:

$$(\mathbb{S}_G)^H = \bigvee_{K<H} \Sigma^\infty BW_K,$$

with W_K the Weyl group of K in H. The homotopy groups of these objects are somewhat accessible (again, at least in principle), but maybe a better idea is to study qualitative information about G-spectra instead.

Question 64. What are some structural properties of the equivariant stable homotopy category?

This connects up to chromatic homotopy theory because there are analogues MU_G of the complex bordism spectrum MU. These aren't really connected to any sort of bordism, but nonetheless they do arise from certain Thom spectra and they do carry sorts of "Chern classes" — and hence they must have some connection to the theory of formal group laws. We don't have full information about $\pi_* MU_G$ the way we do for $\pi_* MU$, so it's hard to say exactly what this connection might be, but we do have some special examples computed. These examples have been worked out by a variety of people: Greenlees, Strickland('s memoir), Križ, Abram, and so on.

Question 65. What does $\pi_* MU_G$ tell us? Generally, does MU_G tell us as much structure in G-spectra as its classical analogue does nonequivariantly?

Of course, MU and chromatic homotopy theory aren't the only possibility for exploring equivariant homotopy theory. Another promising tool are certain decomposition diagrams called "isotropy separation diagrams", which we'll illustrate in the context of C_p-equivariance. A C_p-equivariant spectrum is equivalent data to a spectrum X with a C_p-action, a spectrum Y with no action (thought of as the geometric fixed points of X against C_p), and a map $Y \to X^{tC_p}$ encoding the map to the Tate construction. Specifically, from this information we can reconstruct the genuine fixed points X^{C_p} via the pullback diagram:

$$\begin{tikzcd} X^{C_p} \arrow[r] \arrow[d] & X^{hC_p} \arrow[d] \\ Y \arrow[r] & X^{tC_p}. \end{tikzcd}$$

More generally, we can write complicated isotropy separation diagrams for groups with a more complex subgroup lattice structure. For instance, Σ_3 has four subgroups, and they have to fit together in this world in complicated ways — but it can be made to work. These diagrams play an important role in almost all studies of equivariant theory, and they look rather like a theorem from chromatic homotopy theory:

Theorem 66 (Chromatic fracture). *The category of $E(n)$-local spectra appears as the following coherent pullback:*

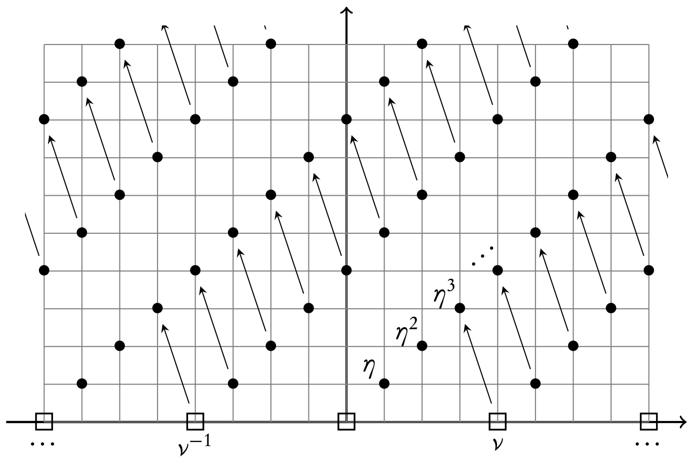
$$\begin{tikzcd} \mathsf{Spectra}_{E(n)} \arrow[r] \arrow[d] & \mathsf{Spectra}_{K(n)} \arrow[d, "L_{E(n-1)}"] \\ \mathsf{Arrows}(\mathsf{Spectra}_{E(n-1)}) \arrow[r, "\text{target}"] & \mathsf{Spectra}_{E(n-1)}. \end{tikzcd}$$

Hence, it's quite possible that we already have the tools in place we need to study this.

Idea 67. In classical chromatic homotopy theory, we mix the primes of $\mathbb{Z}$ with the chromatic primes. Possibly in equivariant stable homotopy theory we'll want to do something analogous by mixing the primes of the Burnside ring of G with the chromatic primes. This is already visible to some extent in our analysis of G-equivariant K-theory K_G.

Idea 68. Humans are generally very bad at thinking about Mackey functors, so isotropy separation often confers a dramatic improvement in perspective.

Here's another seemingly important phenomenon in equivariant theory. We've seen the homotopy fixed point spectra E_n^{hG} for finite subgroups G of the Morava stabilizer group appear a lot this week, especially when talking about TMF, TAF, and most basically KO. The homotopy fixed point spectral sequence computing $\pi_* KU^{hC_2}$ is a well-known exercise:



Each column in this picture computes one homotopy group of KO, and each right-descending diagonal in this picture corresponds to $H^*(C_2; \pi_s KU)$ for a particular value of s. Hence, if we truncate the spectral sequence to get a picture for $\pi_* ku^{hC_2}$, we see that we don't quite compute what you'd expect. There's a scattering of permanent classes in negative degrees with missing incoming differentials, and these are not expected to appear in $\pi_* ko$.

Nonetheless, if you look at the picture, you can see what you'd like to have happen instead: there's a diagonal line on the page across which the classes in positive and negative degrees don't really interact, and if we could truncate along this diagonal line, we'd produce the fixed point spectrum we expect. This also happens when looking at $H^*(\mathcal{M}_{\mathbf{ell}}; \omega^{\otimes *})$ thinking about the homotopy groups of tmf.

The slice filtration takes equivariant phenomena and separates them in an unusual way that often captures exactly this cutoff phenomenon. This in turn seems to suggest that there's some new and unexpected way to look at the G-stable category that we're not quite seeing yet — something important.

Idea 69. There appears to be some deep connection among the slice filtration; nonvertical truncations and gaps in fixed-point spectral sequences; and some unseen third object. This seems like a good project for someone, and whoever's interested should probably first get Vesna's opinion, as she may well have thought about this before.

Remark 70. The slice filtration is built by studying localizing subcategories of G-spectra built out of certain generating objects, just like Postnikov truncations. An extremely important feature of these categories is that they are *merely* localizing and not thick — they are not closed under fiber sequences.

10.2. **Motivic homotopy theory.** Another subject we could try to make an impact on is algebraic geometry, and our most likely target inside of algebraic geometry is motivic homotopy theory. Motivic homotopy shares a lot of large-scale behavior with equivariant homotopy, making it doubly interesting in light of the above discussion. The general idea, of course, is that the stable motivic homotopy category is meant to house "spectra", but given by stabilizing schemes rather than spaces. There are a ton of names attached to this field: Dan Dugger, Dan Isaksen, Kyle Ormsby, Markus Spitzweck, Igor Križ, Po Hu, Morel, Vladimir Voevodsky — and the list goes on almost interminably.

The assimilation of homotopy theory into algebraic geometry via motivic homotopy has already been fairly successful — there are analogues of the Steenrod algebra there, of bordism theories, and so on. The present state of the art is Morel's computation of $\pi_0\mathbb{S}^0$, given by the Grothendieck-Witt ring, which intertwines Milnor K-theory and the Witt ring classifying quadratic forms up to some sort of stable equivalence. All of these ideas have had a serious role to play in the resolution of the Milnor conjecture, which has spurred interest in computing farther out — people are now making headway on computing $\pi_1\mathbb{S}^0$.

What generally makes this complicated is that the motivic homotopy category (like the equivariant stable category and the $K(n)$-local stable category) has a larger Picard object associated to it than the $\mathbb{Z}$ we're accustomed to. In the motivic world, homotopy groups end up being bigraded, generated by a motivic circle and by a simplicial circle. Because of this, computations become exceedingly difficult — a primary complication is that when we commute circles past each other in classical homotopy theory we pick up factors of -1, but motivically there's more than just one circle to commute and more than just one -1.

Remark 71. In principle, motivic computations are supposed to be "easier" than corresponding equivariant computations. For a motivating example, the equivariant cohomology of a point is horrendous, but the motivic cohomology of a point is not. Also, to get you worried, η is neither nilpotent nor torsion in motivic $\pi_*\mathbb{S}^0$.

This all raises a lot of questions to which there are not really even hints of answers — and rather than barging into an unfamiliar field and trying to compute things, let's instead ask a somewhat more reserved question:

Question 72. How can we qualitatively understand motivic stable theory? What are the analogues to understanding things from a chromatic perspective?

Remark 73. The slice filtration discussed in the previous section was actually first investigated motivically.

10.3. **Algebraic K-theory.** Algebraic K-theory provides spectral invariants for module categories — including rings, ring spectra, and generally anything giving rise to a symmetric monoidal category. This has been related in some sense to algebraic topology every since Quillen constructed higher K-theories using homotopical machinery.

We certainly can't compute individual things in algebraic K-theory, and so again we can instead hope to understand general phenomenology connected to the K-theory functor itself. The most strongly related such conjecture is chromatic redshift, which doesn't have a precise statement but is of the rough form:

Conjecture 74 (Chromatic redshift)**.** *The spectrum $K(R)$ has chromatic information of one height higher than that of R itself.*

This is a strange conjecture and it's certainly not something that we understand conceptually, but rather something that's been repeatedly observed in our admittedly limited bunch of computations.

Remark 75. The redshifting element in computations has something to do with the generating cohomological class of BS^1. This appears in the calculation of K-theory in terms of topological Hochschild and cyclic homologies — THH supports an S^1-action, and taking fixed points against it introduces ghost of a BS^1 to the picture. It may also be the case that the B in Quillen's $+$-construction is what's doing the "shifting", since the nonacyclicity of higher Eilenberg-Mac Lanes also has something to do with "seeing higher chromatic data".

Question 76. We could try to understand things from a chromatic point of view. After all, K-theory is closely connected to the functors $R \mapsto BGL_nR$, and so we might ask: is some kind of redshift visible in BGL_nR, BGL_1R, GL_1R, or gl_1R?

Question 77. Since K satisfies a universal property pertaining to sending a module category to its additive invariants, it should be easy to map off of. On the other hand, Lubin-Tate spectra are very nearly the field theories of the stable category, and so they should be easy to map into. Can we in some way detect maps $KR \to E_n$ as R and n vary?

Idea 78. Steve Mitchell has started to study this for $R = HA$ an Eilenberg–Mac Lane spectrum, and his answer is that there are no such maps.

Question 79. Topological cyclic homology is the target of a trace map from K-theory, and its known to capture much of the information present in good situations. So, if we can't understand K-theory, can we instead approach these questions for TC?

On the other hand, there are actual methods available to compute TC, which is a noble goal all its own. This process happens in stages:

(1) Compute THH — this is by now accomplishable by fairly well-established methods.
(2) Use its S^1-equivariant structure to build complicated diagrams involving the Frobenius, restriction, and transfer maps.
(3) Take various sorts of limits and fixed points to produce objects $\mathrm{TR}^n = \mathrm{THH}(R)^{C_{p^n}}$, $\mathrm{TR} = \lim_{\mathrm{res}} \mathrm{TR}^n$, $\mathrm{TF} = \lim_{\mathrm{Frob}} \mathrm{TR}^n$, and $\mathrm{TC} = \mathrm{THH}(R)^{S^1}$.

Question 80 (This is both serious and less serious.)**.** What are some names which abbreviate to TR and TF? Or, what are some better names for these objects?

Of course, TC is not *really* that computable. There are something like $2 \cdot \infty + 2$ spectral sequences involved (stemming from fixed points and from the Tate construction), followed by some additional homotopy limits. What's remarkable is that these intermediate spectral sequences display cutoff phenomena similar to that stemming from the slice filtration.

Question 81. Are the cutoffs in the spectral sequences computing TC related to a slice filtration? Does this inform our understanding of equivariant or motivic stable homotopy theory?

Idea 82. Another approach is to learn even partial information about $\mathrm{THH}(MU)$ or $\mathrm{TC}(MU)$, the "universal" cases. (Various teams of superfriends have done work on / written about this, but there are a lot of moving parts and it's a lot of work.)

Idea 83. Conjectural computations suggest that $\mathrm{THH}(MU)^{hS^1}$ is something like MU again.

Idea 84. Again, the only class that's going to survive this homotopy limit is the class belonging to the circle. This is supposed to be comparable to the computation of $\mathbb{Q} \otimes \lim_n \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p^n$ — a system of v_n-torsion groups can form into a non-v_n-torsion group in the limit.

Idea 85. Is there a connection between THH or TC of MU and that of $\mathbb{S}$, given that one is a nilpotent thickening / Hopf-Galois extension of the other? (Some mentions that Ausoni and Rognes have a lot to say about this.)

10.4. **Transchromatic homotopy theory.** The basic idea here is that we've been studying the $K(n)$-local categories, but we don't have a good conceptual understanding of how these pieces interact. We also heard in Tobi's talk about how little we understand about this interaction with the splitting conjecture.

We don't really study $K(n)$-local spectra from this perspective. Rather, we study their continuous Morava E-theory, i.e., their Morava modules. The relevant object $L_{K(n)}(E_n \wedge X)$ carries an action both of E and of the extended stabilizer group $\mathbb{G}_n$, intertwining in a particular way. This is something like being a module over some kind of spectrum-level group algebra "$E\langle\langle \mathbb{G}_n \rangle\rangle$". Moreover, the Morava module is kind of "complete" as such a module, which should be our lens through which to view the $K(n)$-local category.

So, let's think about sending the spectrum $X_{K(n)}$ to its localization $L_{K(n-1)} X_{K(n)}$ — this should be like sending an $E_n\langle\langle \mathbb{G}_n \rangle\rangle$-module to an $E_{n-1}\langle\langle \mathbb{G}_{n-1} \rangle\rangle$-module.

Question 86. How exactly does this go? What should these symbols mean? How do we put these modules together as n varies?

Hopkins, Kuhn, and Ravenel are interested in this in their paper on character maps, as is Nat with his transchromatic geometry.

10.5. **Multiplicative theory.** We've collectively spent a lot of time thinking about the smash product of spectra, how it gives us commutative ring objects, how it gives us associative ring objects, and all kinds of structures in between. A primary reason to pursue these objects has been computational: having a structured multiplication often induces multiplicative structures on relevant spectral sequences, and this lets you finish computations that you can otherwise merely start.

The downside is that these are very difficult to construct a lot of the time — there are very few spectra that we know have commutative ring structures. Important examples of these that we do know include the Eilenberg–Mac Lane spectrum HR, whose multiplication is essentially unique $\Sigma^\infty_+ \Omega^\infty Y$, which is like the group-algebra $\mathbb{S}[\Omega^\infty Y]$ (though there are a lot of rings which are not group-algebras); and the Thom spectra MO, MU, and Mwhatever. We like MU a lot, and we especially like its relationship to BP — but what about BP?

Question 87. Does BP support a structured commutative multiplication?

This isn't an open problem that gets mentioned much to people outside of our tiny subfield, as it's so strongly rooted in the technical details of the chromatic picture, but it would be extremely helpful to know one way or another. If we had such a structure, the first thing to do would be to try to compute $K(BP)$, which would then be an E_∞-ring which could contain fantastic organizational information. There are at least partial results in this direction:

Theorem 88 (Basterra-Mandell). *BP admits an E_4-structure. (I think they show that the space of such structures is connected but do not show that it is contractible.)*

Theorem 89 (Hu-Križ). *There is no BP-algebra structure on MU.*

Remark 90. So long as I'm speculating, I'd guess that BP does not admit an E_∞-structure. There's little concrete evidence that it should, and in fact some of the essential things we use about BP are known not to mesh with a hypothetical E_∞-structure, which is discouraging. Even so, it would be nice if it were E_∞, but we seem to be hoping for this entirely out of optimism.

It's further worth remarking what this extra structure might buy you:

Theorem 91 (Mandell(, Lurie)). *If R is E_1, then $\mathsf{Modules}_R$ exists (on the level of homotopy categories). If R is E_2, then $\mathsf{Modules}_R$ has a monoidal structure. If R is E_3, then $\mathsf{Modules}_R$ has a braided monoidal structure. If R is E_4, then $\mathsf{Modules}_R$ has a symmetric monoidal structure.*

Remark 92. The E_∞ structure on the objects in HHR is absolutely crucial — the norm map cannot be made sensible without it.

We actually know a way of refining A_∞-structures to E_∞ ones, say via Goerss-Hopkins obstruction theory but also in many other ways. For instance, if BP had such a structure, then the natural map $BP \to H\mathbb{Z}/p$ would be necessarily E_∞, and the action $H_*BP \to H_*H\mathbb{Z}/p$ would govern some amount of what's going on with the Dyer-Lashof algebra. Of course, you can compute this without knowing a priori the existence of such a structure, and then you can feed it into one of these obstruction theories — but things just go south quickly. The way that this goes for MU is to notice that MU is built from BU, which can be resolved by the connective covers $BU\langle 2k\rangle$, which have controllable homology groups. This just doesn't work at all for BP.

Question 93 (Ravenel; mildly unrelated). Find some way to produce spectra from formal group law data.

Remark 94. Hopkins has a "proof" that E_n supports an E_∞-structure straight from the fact that it has an A_∞-structure. This originally ran afoul (pointed out by May) of being too careless about cofibrancy of operads and algebras over operads, and though this has since been fixed no one has written it up. Most interestingly, this does not go through Goerss-Hopkins obstruction theory. On the other hand, somehow both proofs rely heavily on a description of A_∞-maps $E_n \wedge E_n \to E_n$, and Goerss-Hopkins obstruction theory *also* shows that the resulting E_∞-structure is essentially unique.

Theorem 95 (Angeltveit). *The ring spectrum $K(n)$ has a unique $\mathbb{S}$-algebra structure, but not a unique MU-algebra structure.*

Remark 96. The E_∞ properties of completed Johnson-Wilson spectra have been analyzed, but not uncompleted ones. The best result in this direction so far is that $L_{K(2)\vee K(3)\vee\cdots\vee K(n)}E(n)$ is E_∞, but without any kind of completion the answer to the analogous question is not known.

Matt Ando has thought quite a lot about E_∞-orientations of spectra by MU, the intertwining of structured ring spectra with formal group laws. Suppose that R is a so-oriented E_∞ ring spectrum, with an associated formal group law Γ_R. Now, think of a formal group law classified by a map $L \to S$ off the Lazard ring as having a certain shape "R" if it factors as $L \to R_* \to S$ for our fixed map $MU_* = L \to R_*$.

Theorem 97 (Ando). *When R is H_∞-ly MU-oriented (so, in particular, E_∞-ly MU-oriented), then formal group laws of shape R_* are canonically closed under quotients.*

Being a commutative algebra means that you have a map $\mathbb{P}A \to A$, where $\mathbb{P}$ denotes the free E_∞-algebra

$$\mathbb{P}X = \bigvee_{j=0}^{\infty} E^{\wedge j}_{h\Sigma_j},$$

and associativity gives two maps

$$\mathbb{P}\mathbb{P}A \rightrightarrows \mathbb{P}A \longrightarrow A.$$

If these two long composites are equal, then A is said to be E_∞ — if they are merely homotopic, with a specific homotopy, then they are said to be H_∞.

Remark 98. H_∞ is not really operadic in the sense that you might expect. For instance, H_∞ does not imply A_∞ — it doesn't even imply A_4.

Anyway, let's start thinking about Matt's theorem with the example of $\hat{\mathbb{G}}_m$. This p-local formal group has very few subgroups, all of the form $\hat{\mathbb{G}}_m[p^k]$ for varying k. Moreover, there is an isomorphism $\hat{\mathbb{G}}_m/\hat{\mathbb{G}}_m[p^k] \cong \hat{\mathbb{G}}_m$, i.e., a factorization

$$\begin{tikzcd} \hat{\mathbb{G}}_m \arrow[r, "p^k"] \arrow[dr] & \hat{\mathbb{G}}_m \\ & \hat{\mathbb{G}}_m/[p^k]. \arrow[u, dashed] \end{tikzcd}$$

Less locally, we also care about the spectra TMF / Tmf / tmf, all of which admit E_∞ ring structures. They should also fit into this story, and they do: given a subgroup H in the formal completion $\hat{C}$ of an elliptic curve, the formal group $\hat{C}/H$ is actually the formal group associated to the quotient curve itself:

$$\hat{C}/H \cong \widehat{C/H}.$$

This compatibility is in some sense why we get an E_∞ structure on TMF.

Let's also talk about MU-theory — after all, MU is H_∞-ly oriented by the identity, so we should be able to take canonical quotients of any formal group law. This is indeed the case: if $F : L \to R$ is a formal group law over $R[\![x]\!]$, then we get a coordinate $y$on F/H for a subgroup H given by the pointwise formula

$$y = \prod_{\alpha \in H} (x -_F \alpha).$$

(This formula can of course be written with the monic associated to the divisor, rather than in this factored form.) Matt essentially calculated this using Chern classes and a lot of trickery.

What's obvious from this formula is that if you start with something p-typical and an arbitrary subgroup, you have no reason to think that you'll get something p-typical back. This was written up by Johnson–Noel:

Theorem 99 (Johnson–Noel)**.** *The map $MU_* \to BP_*$ classifying p-typical FGLs is not compatible with quotients, and hence cannot be the reduction of an H_∞ map.*

So, we have to find some way of producing such canonical quotients for p-typical FGLs if we expect BP to have an H_∞ structure. Other spectra also show up when trying to analyze this story, about which we can ask similarly difficult questions — namely, $BP\langle n\rangle$ and $E(n)$, with coefficient rings $\mathbb{Z}_{(p)}[v_1, \ldots, v_n]$ and $\mathbb{Z}_{(p)}[v_1, \ldots, v_n][v_n^{-1}]$.

Question 100. Are FGLs of shape $BP\langle n\rangle$ or $E(n)$ closed under quotients? (This would say a lot about the Ausoni-Rognes program.)

What makes this question interesting is that it isn't an entirely formal property. Instead, it depends upon the choice of generators v_i, which are only canonical modulo all the choices of the v_j, $j < i$. Picking "smart" choices of v_i can certainly modify the answer to this question, and it also tells us something about the geometry of the moduli of formal group laws.

Question 101. To what extent does this depend upon the choice of generators v_i?

Idea 102. Andy Baker has thought questions like this before in some paper with "adic" in the title. In it, he rails against having made any choice of coordinates at all in the construction of Johnson-Wilson theory, and he tries to invert all possible v_n rather than a selected one.

Remark 103. There are also $\infty+\infty+2$ cases for which we know that this works. There's $H\mathbb{Z}_{(p)}$ as p varies; there's $BP\langle 1\rangle$ as p varies; there's $BP\langle 2\rangle$ for $p=2$, joint work of Tyler with Niko Naumann; and there's $BP\langle 2\rangle$ for $p=3$, joint work of Tyler with Mike Hill. Neither of these last cases indicate how a proof might go for large primes.

Remark 104. Strickland does some related manipulations with Johnson–Wilson theories in his paper on products on MU-modules, where it's crucial that he work with neither the Hazewinkel generators nor the Kudo–Araki generators, but rather some strange mix of the two. This might be able to be used to prove that there are definitely choices of generators for which this question is answered negatively, at least for MU-modules if not for $\mathbb{S}$-modules.

Idea 105. Certainly this machinery can be used to MU-orient spectra H_∞-ly, and possibly some obstruction theory can then be studied to lift such orientations to E_∞-orientations. There are many examples where this would be interesting, including E-theory. We might also try to compare such approaches to Barry Walker's thesis about p-adic measures versus $K(1)$-local E_∞-orientations. It might also also be worth noting that $E^\vee E$ is vastly nicer from the perspective of commutative algebra than $L_{K(n)}(E_n \wedge_{MU} E_n)$, and that this is a big part of what's standing in the way.

Remark 106. Strickland's subgroup theorem states that

$$\operatorname{Spf} E_n^0 B\Sigma_{p^n} = \operatorname{Sub}_{p^n} \mathbb{G}_{E_n}.$$

This gives even stronger reason to think that power operations are related to finite quotients of formal groups.

11. May 7th: Factoring character maps (Nat Stapleton)

Off and on this semester, we've considered the transchromatic character map

$$E_n(EG\times_G X)\to C_t(EG\times_G \operatorname{Fix}_{n-t} X).$$

We've mentioned the words "twist construction" before, but we haven't really gone into detail. The key theorem concerning the twist construction is that there is a factorization of the transchromatic character map:

$$\begin{tikzcd} E_n(EG\times_G X) \arrow[r] \arrow[dr] & C_t(EG\times_G \operatorname{Fix}_{n-t} X) \\ & B_t(\operatorname{Twist}^G_{n-t}(X)), \arrow[u] \end{tikzcd}$$

which arises from a canonical map $B_t\to C_t$ of rings and a map $EG\times_G \operatorname{Fix}_{n-t} X\to \operatorname{Twist}^G_{n-t} X$ of spaces, where

$$\operatorname{Twist}^G_{n-t}(X) = \hom(*/\!/\mathbb{Z}_p^{n-t}, X/\!/G)/\!/(\mathbb{Q}_p/\mathbb{Z}_p)^{n-t},$$

arising from $(\mathbb{Q}_p/\!/\mathbb{Z}_p)^{n-t}\to(*/\!/\mathbb{Z}_p)^{n-t}$.

This is a concise expression, but it perhaps doesn't explain what's going on. Let me instead try to give concrete models for these objects. First, we have a tried-and-true model for the fix-construction side:

$$EG\times_G \operatorname{Fix}_{n-t}(X) = \coprod_{[\alpha]\in\mathbb{G}_p^{n-t}/\sim} EC(\operatorname{im}\alpha)\times_{C(\operatorname{im}\alpha)} X^{\operatorname{im}\alpha}.$$

We can also form the following pushout of abelian groups:

$$\begin{tikzcd} \mathbb{Z}_p^{n-t} \arrow[r] \arrow[d] & \operatorname{im}\alpha\subseteq G \arrow[d] \\ \mathbb{Q}_p^{n-t} \arrow[r] & \operatorname{im}\alpha\oplus_{\mathbb{Z}_p^{n-t}}\mathbb{Q}_p^{n-t}, \end{tikzcd}$$

leading us to make the definition

$$T(\alpha) := C(\operatorname{im}\alpha) \oplus_{\mathbb{Z}_p^{n-t}} \mathbb{Q}_p^{n-t}.$$

This should give you pause — after all, $C(\operatorname{im}\alpha)$ is not abelian in general — but it is well-defined because of the formula $[g,i] = [g\alpha(z), i+z]$.

By definition, there is a canonical short exact sequence

$$1 \to C(\operatorname{im}\alpha) \to T(\alpha) \to \mathbb{Q}_p/\mathbb{Z}_p^{n-t} \to 1.$$

Recall that $C(\operatorname{im}\alpha)$ acts on $X^{\operatorname{im}\alpha}$, and this is part of the G-action on the Fix construction. This turns out to extend to an action of $T(\alpha)$ on $X^{\operatorname{im}\alpha}$ by the formula $[g,i]\cdot x := gx$ — this follows from

$$[g\alpha(z), i+z]\cdot x = g\alpha(z)\cdot x = g\cdot x.$$

We use this to give a definition of the twist construction:

$$\operatorname{Twist}^G_{n-t}(X) = \coprod_\alpha ET(\alpha) \times_{T(\alpha)} X^{\operatorname{im}\alpha}.$$

This definition makes obvious the map

$$EG \times_G \operatorname{Fix}_{n-t}(X) \to \operatorname{Twist}_{n-t}(X)$$

needed in the character factorization.

It also gives a map

$$\operatorname{Twist}^G_{n-t}(X) \to \operatorname{Twist}^e_{n-t}(*) \simeq B\mathbb{Q}_p/\mathbb{Z}_p^{n-t},$$

which suggests that we try relating it it to the theory of p-divisible groups. In particular, take $G = \mathbb{Z}/p^k$ — then one can calculate

$$E_n \operatorname{Twist}^{\mathbb{Z}/p^k}_t(*) = (\Gamma\mathbb{G}_{E_n^{(BS^1)^{n-t}}} \oplus_{\mathbb{Z}_p^{n-t}} \mathbb{Q}_p^{n-t})[p^k],$$

i.e., the p^k-torsion in the pushout

$$\begin{tikzcd}
\mathbb{Z}_p^{n-t} \arrow[r] \arrow[d] & \mathbb{Q}_p^{n-t} \arrow[d] \\
\mathbb{G}_{E_n^{(BS^1)^{n-t}}} \arrow[r] & \mathbb{G}_{E_n^{(BS^1)^{n-t}}} \oplus_{\mathbb{Z}_p^{n-t}} \mathbb{Q}_p^{n-t},
\end{tikzcd}$$

where $\mathbb{G}_{E_n^{(BS^1)^{n-t}}}$ is determined by the change-of-base pullback square

$$\begin{tikzcd}
\mathbb{G}_{E_n^{(BS^1)^{n-t}}} \arrow[r] \arrow[d] & \mathbb{G}_{E_n} \arrow[d] \\
\operatorname{Spf} E_n^{(BS^1)^{n-t}} \arrow[r] & \operatorname{Spf} E_n.
\end{tikzcd}$$

More explicitly, we can present this ring in coordinates by the formula

$$\prod_{(i_1,\ldots,i_{n-t})\in(\mathbb{Z}/p^k)^{n-t}} E_n[\![q_1,\ldots,q_{n-t}]\!][\![x]\!] / \left([p^k](x) - ([i_1](q_1) +_{\mathbb{G}_{E_n}} \cdots +_{\mathbb{G}_{E_n}} [i_{n-t}](q_{n-t})\right).$$

Finally, note that if all the q_i are set to zero, then we get $\Gamma(\mathbb{G}_{E_n} \oplus \mathbb{Q}_p/\mathbb{Z}_p^{n-t})[p^k]$. This is used to get the relationship between B_t and C_t.[13]

In any event, this is all meant to give a natural factorization of the transchromatic Chern character map through some other ring. That's fairly interesting — where else do we see this?

Question 107. The Bismut-Chern character is a factorization

[13] There was a lot of confusion about whether taking the S^1-action into account changed anything, since it appeared to be acting trivially. Tomer eventually reminded us that if we have a G-spectrum with G decomposing as a normal subgroup N and a quotient G/N, then even if G/N acts trivially, there is a twist taken into account in the G-fixed points by way of the nontriviality of the extension defining G.

$$\begin{tikzcd} & H\mathbb{R}_{S^1}(X) \arrow[d] \\ K(X) \arrow[ur, "\text{Bismut}"] \arrow[r] & H\mathbb{R}(X). \end{tikzcd}$$

Bismut's definition is entirely analytic, which is why we wanted $\mathbb{R}$ rather than $\mathbb{Q}$. Can this map be produced via homotopy theory? (Tomer thinks the answer is yes, and that $\mathbb{R}$ can be replaced by $\mathbb{Q}$ adjoined some periods.)

Idea 108 (Nat). Fei Han has accomplished something like this inside the Stolz-Teichner program.

Question 109. There is a map

$$\mathrm{Twist}^G_{n-t}(X) \to L^{n-t}(X/\!/G) := \hom(\mathbb{Q}_p/\!/\mathbb{Z}_p^{n-t}, X/\!/G)/\!/(\mathbb{Q}_p/\!/\mathbb{Z}_p^{n-t}).$$

Can we build a factorization

$$\begin{tikzcd} E_n(EG \times_G X) \arrow[r] \arrow[dr] & B_t(L^{n-t}(X/\!/G)) \arrow[d] \\ & B_t(\mathrm{Twist}^G_{n-t} X)? \end{tikzcd}$$

Question 110. Both the Fix and the Twist constructions admit actions of $\mathrm{GL}_{n-t}(\mathbb{Z}_p)$. What is the connection between their homotopy orbit spectra and p-divisible groups?

12. May 7th: The E-theory of crossed modules (Tomer Schlank)

Recall that our transchromatic character maps $E_n BG \to C_t L^{n-t} BG$ are so-named because when $n - t = 1$ we have the formula $LBG = \coprod_{g \in G/G} BC(g)$, which is in turn connected to the theory of characters. Furthermore, in the case of $n = 1$, this gives a map $K(BG) \to H\mathbb{C}(LBG)$, which sends a (completed, virtual) representation to its character.

The story of transchromatic characters arises by asking what cohomology theories we can replace K and $H\mathbb{C}$ by, but we could have asked a different question: what else can we use in place of BG? One idea for generalization is that the spaces BG are coincident with pointed homotopy 1-types, and so perhaps we can say something about pointed homotopy 2-types if we can sufficiently express them in terms of group theoretic data. Crossed modules turn out to encode this information well:

Definition 111. A crossed module is a morphism $\delta : M \to P$ of not-necessarily-abelian groups, together with a left-action ${}^P M$ (note that P and M already left-act on themselves by conjugation) such that δ is equivariant in the two ways you could expect:

$$\delta({}^p m) = {}^p \delta(m), \qquad {}^{\delta(m)} m' = {}^m m'.$$

These conditions guarantee that M is normal in P, and so we think of this as the "exact sequence"

$$1 \to \pi_2 X \to M \xrightarrow{\delta} P \to \pi_1 X \to 1.$$

Indeed, there is a functor from crossed modules to 2-types, playing the role of the classifying space functor, which produces a 2-type with these two homotopy groups. On the other hand, one thing we have to be careful about is that many different crossed modules can produce the same 2-type, since M and P are recording something like a presentation of the crossed module. For an easy example, take M mapping to itself by the identity — this has vanishing kernel and quotient, so yields a contractible complex under realization, but obviously M itself can be taken to be nonzero.

A recent theorem of Lurie asserts roughly that character theory works for all π-finite spaces (so, in particular, those arising from finite crossed modules), and our goal is to take this theorem and expand it into computations, ideally identical with those coming from Nat's framework. We would hope to be able to see connections to representation theory, ...

Now, Nat's framework uses groups, so trying to make his theorems work for crossed modules requires going through and checking that we can do the things he does for groups with crossed modules instead. Lots of things do indeed carry over — for instance, the definition of a p-Sylow crossed submodule as p-Sylow subgroups of M and P that map to each other works well. On the other hand, some things don't seem to work, and some things we don't know about. What's also encouraging is that some things from topology carry over as well — we can define the free loopspace of a crossed module as follows.

Definition 112. Select a crossed module $M \to P$ with associated kernel π_2 and quotient π_1. Further select a "basepoint" $[g_0] \in \pi_1$, as well as a lift p_0 of g_0 to P. We define $P[g]$ to be the subgroup of P commuting with p_0. **I must have copied something down wrong...?**

$$M \xrightarrow{m \mapsto (m - {}^{p_0} m, \delta m)} P[g].$$

Remark 113. This is reasonable because the group P is like the paths in the 2-type. So, the new paths in the loopspace are diagrams like

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "p_0"'] \arrow[dr, phantom, "m"] & \cdot \arrow[d, "p_0"] \\ \cdot \arrow[r, "p"'] & \cdot \end{tikzcd}.$$

Two such diagrams paste together as

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "p_0"'] \arrow[dr, phantom, "m"] & \cdot \arrow[r, "p'"] \arrow[d, "p_0"] \arrow[dr, phantom, "m'"] & \cdot \arrow[d, "p_0"] \\ \cdot \arrow[r, "p"'] & \cdot \arrow[r, "p'"'] & \cdot \end{tikzcd},$$

and so one sees where the conjugation condition in the definition arises. **Write something less wishy-washy.**

Idea 114. One thing you could naively hope to see pop up here is a representation theory for 2-vector spaces. However, it seems like this isn't the case, as everyone who looks at this problem for a few weeks decides that 2-vector spaces aren't the way to go — that is, everyone except, perhaps, Ganter and Kapranov. Their work bears investigating.

Idea 115. Another idea is that the Atiyah-Segal theorem asserts that the map from genuine-equivariant K-theory to Borel-equivariant K-theory is described as a completion:

$$K_G(*) = \mathrm{Rep}_G \to \mathrm{Rep}_G^\vee \cong K(BG).$$

The uncompleted object Rep_G includes into $\mathbb{C}^n$, which is some genuinely integral statement, and so one can ask: is there some factorization through the theory of lattices in the setting of crossed modules?

13. May 14th: Higher "q"-expansions and TAF character maps (Sebastian Thyssen)

There's a template story that we care about a lot when trying to get algebraic topology to interact with arithmetic geometry, roughly summarized by the following square:

$$\begin{tikzcd} \text{geometric data} \arrow[r] \arrow[d] & \text{moduli objects} \arrow[d] \\ \text{cohomology theories} \arrow[r] & \text{universal theories.} \end{tikzcd}$$

For instance, this is how people think about TMF:

$$\begin{array}{ccc} \text{elliptic curves} & \longrightarrow & \mathscr{M}_{\text{ell}} \\ \downarrow & & \downarrow \\ Ell_* & \longrightarrow & \text{TMF}. \end{array}$$

and about TAF:

$$\begin{array}{ccc} \begin{array}{c}\text{abelian varieties}\\ + \\ \text{extra structure}\end{array} & \longrightarrow & \text{a Shimura variety} \\ \downarrow & & \downarrow \\ \text{"abelian" cohomology} & \longrightarrow & \text{TAF}. \end{array}$$

There's some extra tickery involved in making this work for higher dimensional abelian varieties; the essential obstacle is that the formal groups arising in algebraic topology are 1-dimensional, whereas those coming from completions of high dimensional varieties will, predictably, be high dimensional. So, to split off a 1-dimensional summand, we're obligated to further consider the following data:

- A principle polarization, which controls A versus its dual A^*. (Note that this is not necessary in the case of elliptic curves, since they are already well-related to their duals.)
- A quadratic, imaginary number field F, so that A carries a complex multiplication through $\mathcal{O}_F$.
- The action of $\mathcal{O}_F$ on A must be of signature $(n,1)$. Loosely speaking this means that $\mathcal{O}_F$ acts on $\operatorname{Lie} A$ by n copies of the natural representation and 1 copy of the conjugate one.

This process eventually yields a cohomology theory, but it is enormously complicated, cf. the memoir "Topological Automorphic Forms" by Mark Behrens and Tyler Lawson. Hence, a question:

Question 116. Can we extract height n information by using lower height theories, obviating the need to compute with TAF directly?

Classically, the answer is yes: the Chern character gives a natural transformation $K(X) \to H\mathbb{Q}(X)$. This is encouraging, and it encourages us to look for more "character maps" of similar form. I'd now like to describe some maps of similar form involving TMF and TAF — but to do so, we'll have to spend some time with number theory first. Let me start by reminding you as to what a modular form is:

(1) A complex, holomorphic modular form of weight k is an analytic function $f : \mathfrak{h} \to \mathbb{C}$ on the upper half-plane satisfying a modular transformation property: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in SL_2(\mathbb{Z})$, we declare
$$f(\gamma \cdot \tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$
(2) Or: a modular form over a ring R of weight k is a rule associating to each elliptic curve $(E/R, d\omega)$ an element $f(E/R, d\omega) \in R$ which is: dependent only upon the isomorphism class of the elliptic curve; homogeneous of degree k in the second variable, i.e., $f(E/R, \lambda d\omega) = \lambda^{-k} f(E/R, d\omega)$; and stable under base change.
(3) Or: a modular form of weight k is a section of $\omega^{\otimes k}$ on $\mathscr{M}_{\text{ell}}$.

All these definitions agree where they interact. Definitions 1 and 2 are evidently equal over $\mathbb{C}$. To tie 2 and 3, note that for an isomorphism $\varphi : E' \to E$, we have $d\omega' = \varphi^* d\omega = \lambda d\omega$ for $\lambda \in R^\times$. Then, we compute
$$\begin{aligned} f(E/R, d\omega) d\omega^{\otimes k} &= f(E'/R, \lambda d\omega') \lambda^k (d\omega')^{\otimes k} \\ &= \lambda^{-k} f(E'/R, d\omega') \lambda^k (d\omega')^{\otimes k}, \end{aligned}$$
and hence this value is invariant under isomorphism.

Over the ring $\mathbb{Z}(\!(q)\!)$, we have the Tate curve — this parametrizes a punctured formal neighborhood on $\mathscr{M}_{\text{ell}}$ of the "missing point at ∞". The q-expansion of a modular form is given by its evaluation on the Tate curve:
$$f \mapsto f(C_{\text{Tate}}, d\omega_{\text{Tate}}) = f_q \in \mathbb{Z}(\!(q)\!).$$

This is said to be holomorphic if it actually lies in the subring $f_q \in \mathbb{Z}[\![q]\!] \subseteq \mathbb{Z}(\!(q)\!)$. Restricting attention to the holomorphic modular forms, this gives a graded ring map

$$\bigoplus_k MF_k \hookrightarrow \mathbb{Z}[u^{\pm}][\![q]\!].$$

In fact, given Lurie's work on building the derived sheaf of elliptic curves, we can instantiate this topologically:

$$\mathrm{Tmf} = \mathcal{O}^{\mathrm{top}}(\overline{\mathcal{M}_{\mathbf{ell}}}) \to \mathcal{O}^{\mathrm{top}}(\mathrm{Spf}\,\mathbb{Z}[\![q]\!]) \text{``=''} KU[\![q]\!].$$

This map is something like a character map, in the sense that it moves from height 2 data to height 1 data.

Remark 117 (Tobi). What are some algebraic characterizations of the image of the q-expansion? Tomer: This is really hard; it involves Hecke operators and many complicated conjectures of Shimura and lots of others. Saul: But, without any extra adjectives, you can write down generators of the ring of modular forms at least. This won't work when you also expect things like level structures or marked points.

Remark 118 (Tobi). Can you also tell us why the map is injective? Sebastian: There are several steps to the argument, but you end up showing that if a modular form vanishes on a formal neighborhood of the cusp, then it's also zero on a dense open, so zero everywhere. The problem appears when you tensor the tautological bundle ω with an arbitrary $\mathcal{O}_F$-module L. The zero locus will still be a dense open, but then you have to actually work to check that it is the whole space.

Remark 119 (Tobi). And what's special about ∞, what if I evaluate somewhere else? Tomer: Well, what other canonical points do you know that are arithmetically global?

Now we move into thinking about automorphic forms, where we might try to reproduce this q-expansion behavior. We'll work with the structure group $U(2,1;\mathbb{Z}[i])$, which contains as a subgroup $SL_2(\mathbb{Z})$ by the embedding

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}.$$

We'll also care about the following generalization of the upper half-plane: $\mathbb{CH}^2 = \left\{(v,w) \middle| \mathrm{Im}\, v - \frac{1}{2} w w^* > 0\right\}$. Given these, an automorphic form φ is an analytic function $\varphi : \mathbb{CH}^2 \to \mathbb{C}$ satisfying modularity: for

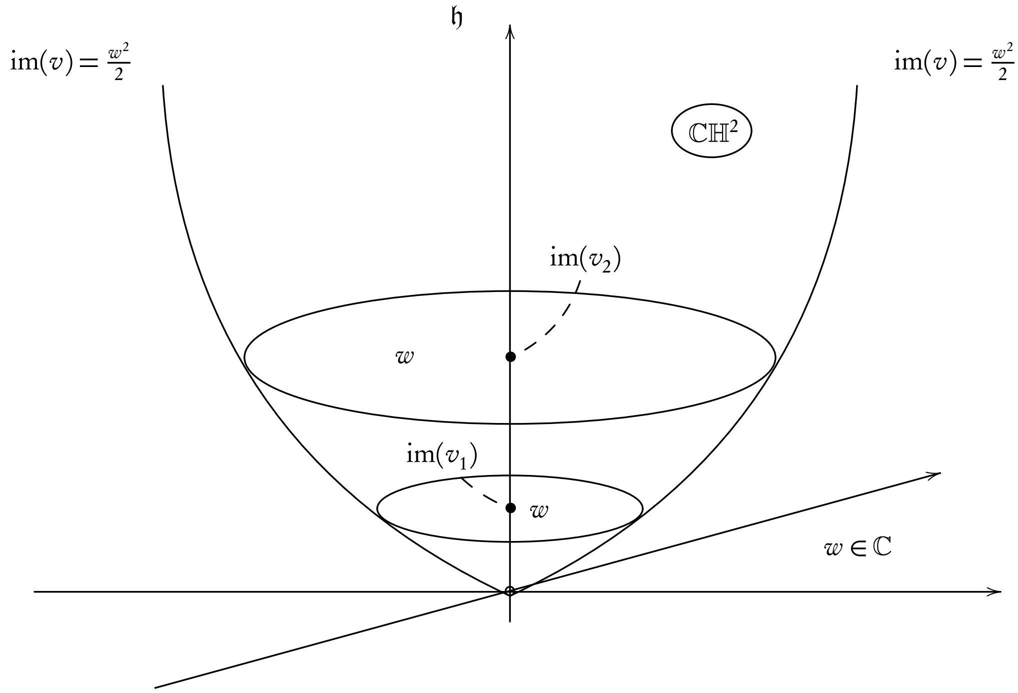
$$\gamma = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

the automorphic form transforms by

$$\varphi(\gamma \cdot (v,w)) = (c_1 v + c_2 w + c_3)^k \varphi(v,w).$$

We have two variables, v and w, in which we can take expansions. The thing number theorists typically do is to take a Fourier expansion in v, which ends up landing in "Jacobi functions$[\![v]\!]$", and there's all sorts of complicated theory that comes out of doing this. The other variable seems rather unexplored; taking Taylor expansion in w at 0 produces something in the ring "Modular forms$[\![w]\!]$".

Let me draw a picture to try to explain why we land in the claimed target ring.



After thinking hard, this forces φ_{n+k} to be a modular form of weight $n+k$, and so begets a map

$$A_k^{U(1,2)} \ni \varphi \mapsto T_{w=0}\varphi \in MF_*[\![w]\!].$$

This can even be made into a map of graded rings, using the same trick as is used for modular forms: a formal sum of two homogeneous forms $\varphi = \varphi_k + \varphi'_{k'}$ can be factored as:

$$\begin{aligned}\varphi_k + \varphi'_{k'} &= \sum_{n=0}^{\infty} \left(\varphi_{n+k}(v) w^n d\omega^{\otimes k} + \varphi'_{n+k'}(v) w^n d\omega^{\otimes k'}\right) \\ &= \sum_{n=0}^{\infty} \left(\varphi_{n+k}(v) d\omega^{\otimes k} + \varphi'_{k'+n}(v) d\omega^{\otimes k'}\right) w^k.\end{aligned}$$

This last object is an element in $MF_*[d\omega][\![w]\!]$, so gives a graded map $A_* \to MF_*[d\omega][\![w]\!]$. Lastly, there are inclusions $\mathbb{CH}^n \to \mathbb{CH}^{n+1}$ for every n, and moreover $\mathbb{CH}^{n-t} \to \mathbb{CH}^{n+1}$ for every n and $t < n$.

Conjecture 120. *There is an expansion principle analogous to the q-expansion principle for elliptic curves.*

The point of such expansion principle is that we can now start of with an, say complex, automorphic form, check that the coefficients of the expansion are say integral and then knew that the automorphic form is already defined over the integers. So assuming that this conjecture holds, the hard work of Behrens and Lawson then produces E_∞-spectra X_n (for n-dim abelian schemes) by evaluating on a suitable formal neighborhood of the expansion locus together with topological analogues of these expansion maps, i.e. (transchromatic) character maps from TAF_{n+1} to the spectra just mentioned!

Idea 121. The target of these higher character maps should yield interesting and unexplored height n theories.

Conjecture 122. *Due to the present construction of the* TAF*-spectra, which is tied to one prime p at a time, we conjecture a good approximation of Morava E-theory by localizing our spectra X_n, i.e.*

$$L_{K(n)} L_p X_n \simeq E_{p,n}.$$

If it wasn't for this dependence on p in the TAF *construction we'd even hope for a more global analogue.*

14. May 24th: Homology of limits and algebraic chromatic splitting (Mike Hopkins)

I was told that this was an informal seminar, and I took that seriously, so this might be a little disorganized — you'll have to bear with me. I want to advertise a purely algebraic problem inside of the Morava E-theory world that I think would be really good to solve. It has to do with comparing two different chromatic layers, and if the calculation comes out a certain way, we'd in particular be able to prove the chromatic splitting conjecture.

This story starts by recalling the homotopy groups of the $E(1)$-local sphere, one of the only computations in chromatic homotopy theory that you can actually do:

$$\pi_* L_{E(1)}\mathbb{S}^0 = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{at } -2 \\ \mathbb{Z} & \text{at } 0 \text{ (for } p \neq 2) \\ \text{other stuff} & \text{at } i \neq 0, -2 \end{cases},$$

where the undescribed groups are finite cyclic groups with the property that

$$\pi_i L_1 \mathbb{S}^0 \otimes \pi_{-2-i} L_1 \mathbb{S}^0 \to \pi_{-2} L_1 \mathbb{S}^0$$

is a perfect pairing. There are lots of different ways of looking at it, including some methods involving p-adic interpolation,[14] and the whole thing behaves as if all the groups want to be captured by that one $\mathbb{Q}/\mathbb{Z}$ in dimension -2. This is even reflected in related computations — the image of j spectrum captures exactly the connective part of this, and if you force that answer to be periodic, then all finite cyclic groups in positive dimensions somehow force the existence of this $\mathbb{Q}/\mathbb{Z}$ in dimension -2.

At higher heights, we can ask the obvious analogues of these questions:

Question 123. Where are the $\mathbb{Q}/\mathbb{Z}$ groups in $\pi_* L_{E(n)}\mathbb{S}^0$? More generally, where are the v_i-divisible groups for $i < n$?

Generally, it's a good idea to try to find pieces of complicated computations which ought to have simple explanations, and then seek those. This feels like one such thing.

Now, there's one other calculation along these lines that we know, and which is actually a little easier: the homotopy of the $K(1)$-local sphere $\hat{L}_{E(1)}\mathbb{S}^0$. This has the property that

$$\pi_* \hat{L}_{E(1)}\mathbb{S}^0 = \pi_* \left(\begin{array}{c} \text{profinite completion} \\ \text{of } L_{E(1)}\mathbb{S}^0 \text{ (at } p) \end{array} \right).$$

Performing this profinite completion on the level of spectra causes the infinite groups to shift around — and you can see this happen when the $\mathbb{Q}/\mathbb{Z}$ in $\pi_{-2} L_{E(1)}\mathbb{S}^0$ becomes a $\mathbb{Z}_p$ in $\pi_{-1} L_{K(1)}\mathbb{S}^0$. So, knowing how the shifts work, we could have instead asked the following question instead and gotten an equivalent answer:

Question 124. What are the groups $\pi_*(L_n\mathbb{S}^0)^\wedge_p \otimes \mathbb{Q}$? When are they nonzero?

These operations can be expressed concisely by studying the Bousfield classes involved — and this explains why the p-completion shows up. We have the following:

$$\begin{aligned} L_n\mathbb{S}^0 &= L_{K(0)\vee\cdots\vee K(n)}\mathbb{S}^0, \\ (L_n\mathbb{S}^0)^\wedge_p &= L_{K(1)\vee\cdots\vee K(n)}\mathbb{S}^0. \end{aligned}$$

These localizations look like they're related, and they are — we have pullback "fracture squares / cubes", an example of which for $n = 2$ is:

$$\begin{array}{ccc} L_{K(1)\vee K(2)}\mathbb{S}^0 & \longrightarrow & L_{K(2)}\mathbb{S}^0 \\ \downarrow & & \downarrow \\ L_{K(1)}\mathbb{S}^0 & \longrightarrow & L_{K(1)}L_{K(2)}\mathbb{S}^0. \end{array}$$

[14] Extremely disappointingly, this was not brought up in this seminar.

So, it would even suffice to understand the rational components of the homotopy groups of these other spectra. The two intermediate corners are something we know — maybe — up to me getting this wrong, at least.[15] This is what I think the theorem is:

Theorem 125 (Maybe)**.** *There is an isomorphism*

$$\pi_* L_{K(n)}\mathbb{S}^0 \otimes \mathbb{Q} \cong H^*(\mathbb{S}_n; \pi_* E_n) \otimes \mathbb{Q} \cong \Lambda[\xi_1, \ldots, \xi_n],$$

with $\xi_i \in \pi_{-2i+1} L_{K(n)}\mathbb{S}^0$.

So, what's left is to understand is the thing in the corner — i.e., $\mathbb{Q} \otimes \pi_* L_{K(n-1)} L_{K(n)}\mathbb{S}^0$. This is part of what the chromatic splitting conjecture is meant to address.

Conjecture 126 ((One sort of the) chromatic splitting conjecture)**.** *Recall the Devinatz-Hopkins equivalence*

$$L_{K(n)}\mathbb{S}^0 \xrightarrow{\simeq} E_n^{h\mathbb{S}_n}.$$

This means that $L_{K(n)}\mathbb{S}^0$ *receives a map* $L_{K(n)}\mathbb{S}^0 \leftarrow (\mathbb{S}^0)^{h\mathbb{S}_n} = D(B\mathbb{S}_n)$ *(where care must be taken with the profinite object* $\mathbb{S}_n$ *in "*$D(B\mathbb{S}_n)$*"). With some work, this begets an element* $\zeta_n \in \pi_{-n} L_{K(n)}\mathbb{S}^0$ *by noticing that the determinant class (or "norm")* $\zeta \in H^1(\mathbb{S}_n; \pi_0 E_n)$ *survives the Adams spectral sequence. Note that* ζ_1 *in* $\pi_* L_{K(1)}\mathbb{S}^0$ *captures the copy of* $\mathbb{Z}_p$ *of interest to us previously. The conjecture is that*

$$L_{K(n-1)}\mathbb{S}^0 \vee L_{K(n-1)}\mathbb{S}^0 \xrightarrow{1\vee\zeta} L_{K(n-1)} L_{K(n)}\mathbb{S}^0$$

is an equivalence. ***Check that I got this last part right.***

Remark 127. This is easy to check at $n = 1$, and elaborate calculations of Shimomura verify it at least for $p \geq 5$. It's generally helpful to have something hold for both of these, essentially because $n = 1$ can't tell the difference between n and n^2, whereas $n = 2$ can.

Remark 128. At some point, people were trying to disprove the telescope conjecture by understanding the locations of the $\mathbb{Q}/\mathbb{Z}$s in the finite localizations via really enormous computational machines. For instance, take X to be an $E(1)$-local finite spectrum. Then we can always find maps to and from large wedges of spheres (by analyzing the rational homotopy type) with the indicated composites:

$$\begin{tikzcd} \bigvee S^n \arrow[r] \arrow[rr, bend right, "p^N"'] & X \arrow[r] \arrow[rr, bend left, "p^N"] & \bigvee S^n \arrow[r] & X. \end{tikzcd}$$

These maps have finite kernel and cokernel, and so knowing where the $\mathbb{Q}/\mathbb{Z}$s are in $\pi_* X$ is identical information to understanding what the wedge of spheres is — that is, it's dependent only upon the rational homotopy type of X. This ought to generalize to v_i-divisibilities as well, and we ought to be able to give qualitative analyses of what's happening.

Through the rest of this talk, we're mostly going to be concerned with two examples. This will save us from dragging indices along everywhere we go.

(1) $n = 1$: We could consider $\pi_* \hat{L}_1\mathbb{S}^0 \otimes \mathbb{Q}$. Now, we know the answer as to where the $\mathbb{Q}/\mathbb{Z}$ groups live, but today we're going to reproduce this same answer in a different way. Now, in place of this, we might try to study $\pi_* K \wedge L_{K(1)}\mathbb{S}^0 \otimes \mathbb{Q}$, where we emphasize that we're using the uncompleted smash product. If we understood that well, then we could easily recover information about the homotopy groups we want. **Is this via an Adams spectral sequence? (That is: why are we studying K-groups?)**
(2) $n = 2$: Similarly, we can try to study the next height up, but let's try to make our decrease in height as small as possible. Namely, we would like to study the groups $v_1^{-1}\pi_* L_{K(2)} M^0(p)$. A simpler replacement for these, as above, are the groups $v_1^{-1}\pi_* E_2 \wedge L_{K(2)} M^0(p)$.[16]

[15]This is in my brain, filed under "things we know." Maybe we know it — or maybe it's just in there because if we could do something really cool with it, then we should be able to go back and figure it out.

[16]Note that because we have a naturally occurring map, we can study the E-homologies of these things to detect equivalences, rather than studying their actual homotopy. Note further that I'm not sure what "map" refers to here.

There's a conjectured answer about all this, which we now endeavor to describe. Begin by considering just the spectrum $L_{K(1)}\mathbb{S}^0$ without the extra inversion step, and recall that $L_{K(1)}\mathbb{S}^0 = \lim_m L_1 M(p^m)$. We want to understand $K_* L_{K(1)}\mathbb{S}^0$, and we know that $K_* L_1 M^0(p^m) = K_*/p^m$. So, what's left is a question about interchanging K_*-homology with inverse limits. This is also how things go in our program at $n=2$: we want to study the system

$$L_{K(2)}M^0(p) = \lim L_2 M^0(p) \wedge M^0(p^{n_0}, v_1^{n_1}) = \lim M^0(p) \cup_{v_1^{n_1}} C\Sigma^{n_1|v_1|}M^0(p).$$

Analogously, the intermediate homologies of the spectra in this system are

$$(E_2)_* L_2 M^0(p) \wedge M^0(p^{n_0}, v_1^{n_1}) = (E_2)_*/\langle p, v_1^{n_1}\rangle.$$

This all begets the following natural question:

Question 129. How do we calculate the E_n-homology of an inverse limit? Can we do this for $E(n)$-local systems?

Remark 130. You can see that there's something to this question in an example. There's an equivalence $M(p) = \lim M(p) \wedge M(p^{n_0}, v_1^{n_1})$, but all the terms in the right-hand side are K_*-acyclic, while the left-hand side is not. However, if we $K(2)$-localize appropriately, things get a lot better, and that's the miracle we're going to discuss.

Let's discuss this situation further: set $E = E_n$ and consider a system

$$\cdots \to X_m \to X_{m-1} \to \cdots,$$

where all the spectra are local for $E(n)$. Maybe the E-homology of the inverse limit of this system is complicated, but the homotopy is not so bad: there's a Milnor short exact sequence

$$0 \to \lim{}^1 \pi_{*+1} X_m \to \pi_* \lim X_m \to \lim \pi_* X_m \to 0,$$

arising from the long exact sequence associated to the fibration

$$\lim X_m \to \prod_m X_m \xrightarrow{\text{shift}} \prod_m X_m.$$

So, it's actually an equivalent problem to understand the E-homology of products.

For now, let's suppose that our system is such that $\lim^1$ vanishes — this hypothesis is definitely inessential (in fact, that will sort of end up being the point), but maybe for now it will be simplify the exposition. Anyway, it would be great if we had a formula for E_*X in terms of π_*X, but of course we don't in general. There is one special case where we do: if X is an E-module, then there is an isomorphism $E_*X \cong E_*E \otimes_{E_*} \pi_*X$. So, if our diagram was a tower of E-modules with E-module maps, we could put these facts together[17] to get

$$E_* \lim X_m = E_*E \otimes_{E_*} \lim \pi_* X_m.$$

This formula shows that there is not an isomorphism $\lim E_*X_m \ncong \lim E_*E \otimes_{E_*} \pi_*X_m$. That's not surprising, as we know in our hearts that homology doesn't commute with inverse limits, and this is exactly the phenomenon we're trying to capture. However, we can actually make this statement true if we make our symbols mean something else — specifically, this statement is false *if we understand the limit to be in the category of graded groups*. The functor $E_*E \otimes_{E_*} -$ suggests that we really mean something else, as it's the functor which sends an E_*-module to the cofree E_*E-comodule over it. This is right-adjoint to the forgetful functor:

$$\mathsf{Comodules}_{E_*E} \underset{E_*E \otimes_{E_*} -}{\overset{\text{forget}}{\rightleftarrows}} \mathsf{Modules}_{E_*}.$$

Since it's a right-adjoint, it commutes with limits, and so we see that the thing we're computing *is* a limit in the category of E_*E-comodules — and really we could have been doing everything in the category of E_*E-comodules, including the Milnor sequence and so on.

Remark 131. Note that we mean E_*E-comodule in the most literal sense — i.e., these things are *not* equivalent to $\mathbb{S}_n$-modules. That becomes true only after I_n-adic completion. The difference between these two objects is recorded in the coaction map: there is a map $\psi : M \to E_*E \otimes M$ with $\psi(m) = \sum b_i \otimes m_i$, which is necessarily a *finite* sum. However, a naive infinite limit could potentially convert this into an *infinite* sum. The limit in E_*E-comodules is exactly the E_*-submodule on which ψ is given as a finite sum. This makes computations exceedingly awkward, but it gives some small bit of intuition as to what's going on.

[17] This statement is still ignoring $\lim^1$. If we wanted that too, we could re-state this in terms of products, and everything would work out.

We can drop the hypothesis that we're working with E-module spectra by noting that every spectrum admits a cosimplicial resolution in terms of E-modules:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
X_m \arrow[r] \arrow[d] & E \wedge X_m \arrow[r] \arrow[d] & E \wedge E \wedge X_m \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \cdots \\
X_{m-1} \arrow[r] \arrow[d] & E \wedge X_{m-1} \arrow[r] \arrow[d] & E \wedge E \wedge X_{m-1} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

We now have access to the E-homologies of all the terms in these resolutions. This is still hard of course — we seem to have traded one inverse limit for another. Luckily, we have a nilpotence-era theorem to help us out:

Theorem 132 (H., Ravenel)**.** *Any $E(n)$-local spectrum X has a finite E-resolution. (In fact, doing this for the sphere and smashing through shows that every $E(n)$-local spectrum has a finite resolution of uniform length.)*

Theorem 133. *If $\{X_\alpha\}$ is an inverse system of $E(n)$-local spectra, then this process begets a spectral sequence*

$$\lim^s_{\mathsf{Comodules}_{E_*E}} E_* X_\alpha \Rightarrow E_{*-s} \lim X_\alpha.$$

No one has learned to calculate with this spectral sequence, but it doesn't feel like this is because it's especially hard — just that no one has really resolved to sit down and do it. In fact, my strong feeling is that this is a fundamental thing that transchromatic people have missed out on. We ought to be able to compute more about these derived functors of the inverse limit, and managing to do so would open a lot of doors.

Let's try it out in our easy, concrete example of $n = 1$. The first thing we'll need is a (finite) resolution of $L_{E(1)} M^0(p^m)$. This comes from the usual fiber sequence

$$L_{E(1)} \mathbb{S}^0 \cup_{p^m} e^1 \to K/p^m \xrightarrow{\psi^\ell - 1} K/p^m,$$

where ℓ is a topological generator of $\mathbb{Z}_p^\times$ and $p \neq 2$. Applying K_* and taking the limit in K_*K-comodules gives[18]

$$\lim_{\mathsf{Comodules}_{K_*K}} K_*K/p^m = K_*K.$$

Hence, this gives an exact sequence

$$0 \to \lim_{\mathsf{Comodules}_{K_*K}} K_*/p^m \to K_*K \xrightarrow{\psi^\ell - 1} K_*K \to \lim^1_{\mathsf{Comodules}_{E_*E}} K_*/p^m \to 0.$$

We're supposed to be able to understand this algebro-geometrically, without picking a basis (or, as they say, "coordinates"). We needed to feed in information about the stabilizer group and its action in homotopy theory to get here, but once we're here we should be able to hand everything over to algebra. To start, K is Landweber flat, which means we have an isomorphism $\operatorname{Spec} \pi_0 K \wedge K = \operatorname{Aut} \hat{\mathbb{G}}_m$. The map ψ^ℓ induces a map $\operatorname{Aut} \hat{\mathbb{G}}_m \to \operatorname{Aut} \hat{\mathbb{G}}_m$, which acts by $f \mapsto [\ell] \circ f$. Now, we're really interested in the rationalization of these groups $\mathbb{Q} \otimes K_*K$, and rationally we have access to a logarithm. With some work, this gives $\mathbb{Q} \otimes \pi_0 K \wedge K \cong \mathbb{Q}[b^\pm]$, where b is given by the ratio of the v_1 elements from each factor, and the action of ψ^ℓ is given by $\psi^\ell(b) = \ell \cdot b$. **It would be nice to flesh out the formal geometry.** This allows us to compute the inverse limit exact sequence:

[18] Recall that we're working with $K = E_1 = K_p^\wedge$, which is p-adically complete — but that we're *not* completing any of our smash products.

$$\begin{gathered}
\mathbb{Q} \longrightarrow \mathbb{Q} \otimes \pi_0 K \wedge K \longrightarrow \mathbb{Q} \otimes \pi_0 K \wedge K \longrightarrow \mathbb{Q} \\
b^n \longmapsto (\ell^n - 1) b^n \\
b^0 = 1 \longmapsto (\ell^0 - 1) b^0 = 0.
\end{gathered}$$

Tensoring up to K_*-coefficients gives what we were after.[19]

We can also imagine how this might go for $n = 2$. The spectra in our system there were $X_m = L_{E(2)} M^0(p) \cup_{v_1^m} C\Sigma^{m|v_1|} M^0(p)$, and we want some resolution

$$X_m \to E_2 X_m \to \cdots,$$

which should somehow involve the structure of $\mathbb{S}_2$. We use our program to calculate the levelwise E-homology:

$$E_* \lim X_m \to \lim_{\mathsf{Comodules}_{E_* E}} E_* E \wedge X_m \rightrightarrows \cdots.$$

Finally, we invert v_1 and study the cohomology of the resulting resolution. Just as in the example of $n = 1$, we're supposed to calculate that the various maps act by 0 on the element 1, and this is what produces the ζ_n element we're after. We summarize all this for arbitrary height in the following conjecture:

Conjecture 134 (Algebraic chromatic splitting conjecture)**.**

$$\lim_{\mathsf{Comodules}_{E_* E}}{}^i E_* / \langle p, \ldots, v_{n-2}, v_{n-1}^m \rangle = \begin{cases} 0 & \text{for } i > 1, \\ v_{n-1}^{-1} E_* / \langle p, \ldots, v_2 \rangle & \text{for } i = 0, 1. \end{cases}$$

(This would verify the ordinary chromatic splitting conjecture.)

Remark 135. It's unclear whether this should depend upon the size of the ambient prime. Mike's gut feeling is that it shouldn't, but Shimomura claimed that the ordinary chromatic splitting conjecture was false at small primes, though this has been re-contested. The web is sufficiently tangled that it's hard to really tell. If anything goes wrong, it's that the infinite cohomological dimension of the stabilizer group somehow complicates the resolutions we need to use.

Remark 136. I'd really like to state all this in terms of the Morava stabilizer group, though admittedly this may just be out of personal computational prejudice. This is really hard, of course — we're relating $\mathbb{S}_n$ to $\mathbb{S}_{n-1}$ — and the "computability" of these derived inverse limit functors is also part of their appeal. One thing that's arised since then but which we weren't originally thinking about is the interaction with Barsotti–Tate groups, specifically the idea that height reduction isn't losing height but rather pushing it into an étale component.

Remark 137. At some point in the distant past, the Ext groups of an arbitrary ring were viewed as impossible to compute, but then Tate and all his friends came along and gave these beautiful methods for computing Ext in certain cases. As you go along and read about these things, you start to build up a library of things you do understand, and by the end of it you realize you can actually compute quite a lot. One of the major goals of this sort of project would be to start building such a library — to find simple examples of diagrams whose derived inverse limits we can compute, then to begin some systematic approach to classes of examples.

Remark 138. Shimomura's calculations of the homotopy of the $E(2)$-local sphere verifies (again, at least at large primes) that the ordinary chromatic splitting conjecture holds. His machine, however, is completely algebraic, since the Adams spectral sequence that would typically follow the chromatic spectral sequence is collapsing. So, while I haven't checked this, it's hard to imagine that his calculation somehow avoids checking the algebraic version of the chromatic splitting conjecture as well.

Remark 139. Paul Goerss gave a talk at some point called *The Homology of Inverse Limits*, and Hal Sadofsky claimed to have a bunch of computations related to this. I'm not sure what happened to either of their projects, but they might be good people to talk to if you want to hear more about these things or if you have something to share yourself.

[19] The differential db/b in H^1 is supposed to be important to this, and it's also supposed to indicate how a lot of this proceeds in general.

Appendix A. Further projects

Some projects we mentioned through the semester seemed interesting, but were either too underdeveloped to speak about or we simply ran out of time. It seems remiss to omit their mention from these notes entirely, so here they are.

A.1. Other things involving p-divisible groups. A broader program that could be investigated is to instantiate other entries from the chart in Section 1.3 in terms of p-divisible groups and height modulated Morava E-theory. Work in progress includes:

- Nat and Tobias have largely worked out a study of level structures on the group $\pi_0 L_{K(t)} E_n \otimes \mathbb{G}_{E_n}$ and their instantiation in algebraic topology.
- Nat has produced an analogue of the result that $BU(m)$ represents the divisor scheme $\mathrm{Div}_m^+(\mathbb{G}_{E_n})$ by studying the p-torsion points of $U(m)$.
- Nat has also begun investigating a p-divisible analogue of Strickland's theorem that $B\Sigma_m$ represents the scheme of subgroups $\mathrm{Sub}_m(\mathbb{G}_{E_n})$.

This leaves many constructions untouched, however. For instance, isogenies of formal groups are known to be connected to power operations, by work of Matthew Ando; this is connected to the third point above. Matt, along with Mike Hopkins and Neil Strickland, have also shown that the formal scheme associated to $BU\langle 6\rangle$ is connected to Θ-structures (or cubical structures) in the sense of Lawrence Breen or David Mumford. Less classically, Neil Strickland's student Sam Marsh computed the E-theory of various linear algebraic groups of finite fields, and while he didn't phrase his results in this language the answer appears to be related to symmetric powers of formal schemes. An analysis of any of these ideas or of carrying over any other algebro-geometric description of a space using classical Morava E-theory to height-modulated Morava E-theory would be very interesting.

A.2. Morava E-theory at $n = \infty$. Several of us have wondered aloud at one point or another what the Morava E-theory associated to the degenerate point $\hat{\mathbb{G}}_a \in \mathscr{M}_{\mathrm{fg}}$ might look like. This seems hard to make sense of, and possibly someone with a superior understanding of the moduli stack would warn us off of trying to study it seriously, but nevertheless there has been idle speculation.

A.3. E_∞-presentations of MU. (This is an idea of Matt Ando's. He has a graduate student, Nerses Aramyan, who may be working on this project or something extremely close. Those intrigued by this would do well to talk to him first, so that no one steps on his toes.)

There are not many spectra for which we have E_∞-structures, and there are even fewer for which we have E_∞-maps among them. Some of these (e.g., MU) come from geometric considerations, with which it is typically very hard to do algebraic calculations, and some of these (e.g., E_n) come from very hard algebraic calculations alone, for which no geometric analogue is known.

There is one result which suggests an algebraic program that might be used to calculate E_∞-MU-orientations. A complex oriented cohomology theory can be defined in two ways: as a ring spectrum E receiving a ring map $MU \to E$, or as a ring spectrum E together with a class $x \in E^2\mathbb{CP}^\infty$ restricting to the unit under

$$E^2\mathbb{CP}^\infty \to E^2\mathbb{CP}^1 \cong \tilde{E}^0\mathbb{S}^0.$$

This second definition can also be expressed diagrammatically, using the identification of the Thom spectrum of the tautological bundle over $\mathbb{CP}^n$:

$$T(\mathscr{L} - 1 \downarrow \mathbb{CP}^n) = \Sigma^{-2+\infty}\mathbb{CP}^{n+1}.$$

Adam's observation is that the choice of such a cohomology class x is equivalent to a choice of factorization of the unit map:

$$\begin{tikzcd} T(\mathscr{L} - 1 \downarrow \mathbb{CP}^0) \simeq \Sigma^{-2}\mathbb{CP}^1 \arrow[r, "\simeq"] \arrow[d] & \mathbb{S}^0 \arrow[d, "\eta"] \\ T(\mathscr{L} - 1 \downarrow \mathbb{CP}^\infty) \simeq \Sigma^{-2}\mathbb{CP}^\infty \arrow[r, dashed, "x"] & E. \end{tikzcd}$$

Since MU is canonically complex-oriented, this begets a map $T(\mathscr{L} - 1 \downarrow \mathbb{C}\mathrm{P}^\infty) \to MU$, which on rational homology is the unit map

$$\mathbb{Q}\{\beta_1, \beta_2, \ldots\} \to \mathbb{Q}[b_1, b_2, \ldots].$$

Rationally, the free E_∞-ring-spectrum $\mathbb{P}(X)$ on a spectrum X has known action on homology:

$$H\mathbb{Q}_*\mathbb{P}(X) = \operatorname{Sym} \widetilde{H\mathbb{Q}_*} X.$$

Hence, the natural map $\mathbb{P}T(\mathscr{L} - 1 \downarrow \mathbb{C}\mathrm{P}^\infty) \to MU$ is a rational equivalence.

Thinking of rational information as "chromatic height 0", one might then ask what happens at chromatic height 1, i.e., whether this map is a $K(1)_*$-equivalence. It isn't — the E_1-Dyer-Lashof algebra acts freely (unsurprisingly) on the free E_∞-ring-spectrum, but its action is not free on MU. However, Matthew Ando believes that work of McClure and of Ando–Hopkins–Strickland on H_∞-MU-orientations suggests how this situation might be repaired:

$$\begin{tikzcd}
& & C \arrow[d, dashed] \\
\mathbb{P}T(\mathscr{L} - 1 \downarrow \mathbb{C}\mathrm{P}^\infty) \arrow[rru, "\text{coeq}"] \arrow[rr] & & MU \\
\mathbb{P}T(\mathscr{L} \otimes \overline{\mathrm{Perm}} \downarrow \mathbb{C}\mathrm{P}^\infty \times B\Sigma_p). \arrow[u, shift left] \arrow[u, shift right] & &
\end{tikzcd}$$

The spectrum $T(\mathscr{L} - 1 \downarrow \mathbb{C}\mathrm{P}^\infty)$ should receive a pair of maps: one corresponding to a sort of trivial inclusion of a wedge summand, and another corresponding to a sort of norm construction. These should agree upon postcomposition to MU, and so produce a map off of their coequalizer C in E_∞-ring-spectra. This map is conjectured to be a $K(1)$-equivalence, and since $B\Sigma_p$ is rationally acyclic, an $E(1)$-equivalence.

This has really pleasant implications — using the adjunction

$$E_\infty\text{-}\mathsf{RingSpectra}(\mathbb{P}X, Y) \simeq \mathsf{Spectra}(X, Y)$$

we produce from the coequalizer sequence a fiber sequence of mapping spaces:

$$\mathsf{Spectra}(T(\mathscr{L} \otimes \overline{\mathrm{Perm}} \downarrow \mathbb{C}\mathrm{P}^\infty \times B\Sigma_p), Y) \leftleftarrows \mathsf{Spectra}(\Sigma^{-2}\mathbb{C}\mathrm{P}^\infty, Y) \leftarrow E_\infty\text{-}\mathsf{RingSpectra}(MU, Y),$$

where Y is an $E(1)$-local E_∞-ring-spectrum. This gives a concrete condition for checking when a given coordinate on a $K(1)$-local complex-orientable spectrum is in fact an E_∞-coordinate — it simply has to pull back to the same element of cohomology under two maps.

More generally, one would hope that a similar resolution (perhaps involving $B\Sigma_{p^2}$) could be used to study at least $E(2)$-local E_∞-MU-orientations, if not $E(n)$-local ones — these may have been recently made accessible by Charles Rezk's work on the Koszul-ality of the Dyer-Lashof algebra for $E(n)$. If this works out sufficiently well, it may also give access to a proof that the $MU\langle 6\rangle$-orientation of tmf (which is $E(2)$-local) is E_∞. Other related work includes the thesis of Jan-David Möllers, a student of Gerd Laures, and of Barry Walker, a student of Charles Rezk. It's also possible that the obstruction theory of Niles Johnson and Justin Noel is related — these resolutions may be smart, short resolutions of their general procedure. There is also a sequence due to Arone and Lesh, in a paper titled *Filtered spectra arising from permutative categories*, which appears to at least be superficially related to this problem.

A.4. **Iwasawa theory.** (All of this is an idea of Mike Hopkins.)

Consider, again, the homotopy groups of the $K(1)$-local sphere. For $p \geq 3$ the $K(1)$-local Adams spectral sequence collapses, so the answer is entirely computed by the chromatic spectral sequence. The chromatic spectral sequence is concentrated in two lines: $H^{0,*}(\mathbb{S}_1; E_1)$ is a single $\mathbb{Z}_p$ in degree 0, and $H^{1,1+*}(\mathbb{S}_1; E_1)$ contains a scattering of groups, collectively called the α-family, which altogether are given by the following formula:

$$\pi_n L_{K(1)}\mathbb{S}^0 = \begin{cases} \mathbb{Z}_p & \text{when } n = 0, \\ \mathbb{Z}_p/(ps) & \text{when } n = s|v_1| - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Chromatic homotopy theory gives us powerful methods for computations, but any geometry present in the system exists integrally, and so it is frequently useful to try to produce integral models for the phenomena we see when working at a family of chromatic primes. The denominators in this formula are, in fact, encoded by a familiar integral family: the Bernoulli numbers. Specifically, we have

$$\pi_{s|v_1|-1} L_{K(1)} \mathbb{S}^0 = \mathbb{Z}_p / \operatorname{denom}(B_{2s}/2s).$$

Here's a second interesting observation: the expression $\mathbb{Z}_p/(ps)$ can be written in terms of the p-adic valuation as $\mathbb{Z}_p/p^{1-|s|_p}$, which is evidently p-adically continuous in s. It's also evident that we can replace the integer $s \in \mathbb{Z}$ by a p-adic integer $s \in \mathbb{Z}_p$ and the formula will still give a result. In fact, this isn't a totally insane idea: the even part of the Picard group of the $K(1)$-local category is given by $\mathbb{Z}_p^\times$, and so we could conceive of such homotopy groups graded against p-adic integers. Further computation reveals that this guess at the extension of our formula is actually correct — the homotopy groups graded over the p-adic spheres simply p-adically interpolates the standard $K(1)$-local homotopy groups.

Moved by this, we can ask for a continuous object in which the Bernoulli numbers appear — and number theory has gifted us with one of those too: the Riemann ζ-function, where the Bernoulli numbers appear as the special values

$$\zeta(1-2s) = \frac{B_{2s}}{2s}.$$

This is neat, and it gives us an idea of where to go, but the integral homotopy groups are inappropriate for studying continuous phenomena. After all, we don't know of such a thing as $\mathbb{C}$-graded homotopy, and we already have a notion of p-adically graded homotopy in the $K(1)$-local setting. So, perhaps we're after a p-adic analogue of the ζ-function.

Number theorists have also described such an object, though they have decided that it resides in the field of Galois theory, in a further subfield now called Iwasawa theory. Define the group G to be the Galois group of the maximal cyclotomic extension: $G = \operatorname{Gal}(\mathbb{Q}(\zeta_{p^\infty}) : \mathbb{Q})$. This comes with a natural character $\varkappa : G \to \mathbb{Z}_p^\times$, determined by the formula

$$\zeta_{p^n}^{\varkappa(\sigma)} = \sigma \cdot \zeta_{p^n}$$

for any $\sigma \in G$. To build a ring-theoretic source of inputs to $\varkappa$, we construct the profinitely-continuous group-ring $\Lambda = \mathbb{Z}_p[\![G]\!]$, sometimes referred to as "the Iwasawa algebra." The ring Λ is to be thought of as the ring of holomorphic functions, and the pushforward map $\varkappa_*^{2r} : \Lambda \to \mathbb{Z}_p$ as evaluation at r. Then, the p-adic ζ-function is defined as a certain element of the fraction field of Λ (i.e., as a "meromorphic function") which interpolates the complex zeta function at our chosen points and which has dramatic constraints on its poles.

This is a lot to swallow, and it's not clear how to connect this set-up to chromatic homotopy theory exactly. Nonetheless, Neil Strickland[20] has forged ahead and rephrased our computation of the homotopy groups of the $K(1)$-local sphere in Iwasawa-theoretic terms. Recall that $\operatorname{Pic}_1^{\text{even}} \cong \mathbb{Z}_p^\times$, which we interpret as $\mathbb{Z}_p^\times = \operatorname{End}(\mathbb{Z}_p^\times)$ by the correspondence between the function $\beta : x \mapsto x^b$ and b. Now take G to be $G = \operatorname{Pic}_1^* = \hom(\operatorname{Pic}_1^{\text{even}}, \mathbb{Z}_p^\times)$, so that $\Lambda = \mathbb{Z}_p[\![G]\!]$. We construct an "Iwasawa module" $\pi\mathbb{S}^0_{K(1)}$ over Λ, along with auxiliary modules M_β for each $\beta \in \operatorname{Pic}_1$ which altogether have the relation

$$\pi\mathbb{S}^0_{K(1)} \otimes_\Lambda M_\beta = \pi_\beta L_{K(1)}\mathbb{S}^0.$$

Here's how he does it: he takes $\pi\mathbb{S}^0_{K(1)}$ to be $\mathbb{Z}_p$ with the trivial Pic_1^*-action, and he makes M_β to be $\mathbb{Z}_p$ with the action $\lambda \cdot x = \lambda(\beta)x = \lambda^b x$. We can check that this gives the right answer via the following calculation, taking $x \otimes y$ to be a generator of the above tensor product and b to be a topological generator of $\mathbb{Z}_p^\times$:

$$\begin{aligned} x \otimes y = (\beta_b x) \otimes y = x \otimes (\beta_b y) &= x \otimes \lambda^b y \\ &\Rightarrow 0 = (1-\lambda^b)(x \otimes y). \end{aligned}$$

This presents us with two operations: in the case $\lambda = 1 + ps$ this produces the quotient $\mathbb{Z}_p/(ps)$, and in every other case the coefficient is a unit and the tensor product collapses to the zero module.

This was abandoned after a few people thought really hard about the $K(2)$-local case, found it to be very difficult, and then got distracted by much more attractive and modern developments in homotopy theory. However, Mark

[20]More exactly, Mike says that this is his model, but the only place it appears in the literature is in some published notes of Neil's.

Behrens published a result a few years ago that indicates there may yet be hope for a $K(2)$-local statement. Mark's result concerns the organization of the β-family, which are a bunch of analogous elements in $H^{2,*}$ at chromatic height 2, using (spectral related to) TMF. If TMF is involved, then you should expect their existence and nonexistence to be reflected and recorded by statements about modular forms, and this is exactly the case: the element $\beta_{i/j,k}$ exists exactly when a corresponding modular form $f_{i/j,k}$ exists, satisfying some minimality property and a variety of complicated and confusing conditions (even for seasoned number theorists) about the weights of f and its reductions mod powers of p.

What might this mean in the context of the above program? Remember that in the end this all came down to picking off the denominator of some Bernoulli numbers, which is supposed to tell you how many times you can divide your favorite α-element by p. To phrase this operation algebraically, you'd want to follow the composite

$$\mathbb{Z}_p \xrightarrow{\zeta} \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p,$$

which detects exactly the fractional part of the p-adic rational special value of the ζ-function. Analogously, the divisibility properties of modular forms are encoded in the theory of L-functions. Moreover, there's even an analogous quotient: L-functions are valued in modular forms, and p-adic modular forms are essentially built out of integral modular forms along with reciprocals of the Eisenstein series. Hence, one could hypothesize the existence of a fantastical p-adic L-function whose composite

$$\text{unknown source} \to p\text{-adic } MF \to \frac{p\text{-adic } MF}{\text{integral } MF}$$

records the divisibilities in the β-family.

That's already a lot to ask for — but, ideally, this would all fit into an Iwasawa theory for Pic_2, which would wrap up the homotopy groups of $L_{K(2)}\mathbb{S}^0$ into a single Iwasawa module. You'll notice that the module $\pi\mathbb{S}^0_{K(1)}$ was extraordinarily simple. This is partly because the actual homotopy groups are not so awful, but it's also reflective of a broader phenomenon in number theory: computing special values of modular forms, L-functions, or whatever else is potentially very difficult, but the objects themselves are often easier to deal with "in the aggregate." By the same token, one could hope for a sort of L-function which "embodies" the $K(2)$-local homotopy groups of spheres in a way that gave interesting structural information about them without requiring us to unpack the Λ-module into the individual groups. Such a result would be a really extraordinary victory for stable homotopy theory.