MA 472 G Problem Sessions

Problem Session 12, Apr 24

Recall the Lebesgue Number Lemma: If (X, d) is a compact metric space and $\mathcal{U}$ is an open covering of X, then there is a number $\delta > 0$ such that every set $S \subset X$ of diameter $\operatorname{diam}(S) < \delta$ is contained in a member U of $\mathcal{U}$.

Problem 1: Uniform Continuity. If (X, d) is a compact metric space and $f : X \longrightarrow \mathbf{R}$ is a continuous function, use the Lebesgue Number Lemma to prove that f is uniformly continuous.

SOLUTION: Let $\varepsilon > 0$. Cover $\mathbf{R}$ with open sets of diameter $< \varepsilon$. For example, we can take $\mathcal{V} = \{V_a \mid a \in \mathbf{R}\}$, where $V_a = (a - \varepsilon/3, a + \varepsilon/3)$. Since f is continuous, $U_a = f^{-1}(V_a)$ is an open set in X. Clearly $\mathcal{U} = \{U_a \mid a \in \mathbf{R}\}$ is an open covering of X. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$ and $x, y \in X$ be such that $d(x, y) < \delta$. Let $S = \{x, y\}$. Then $\operatorname{diam}(S) < \delta$ and therefore $S \subset U_a$ for some $a \in \mathbf{R}$. Then $f(S) = \{f(x), f(y)\} \subset V_a$ and therefore $|f(x) - f(y)| < \varepsilon$. Thus f is uniformly continuous. ★

Problem 2: Bounded Functions and Compactness. Let $A \subset \mathbf{R}$ be any set. If A is compact, then the Extreme Value Theorem implies that every continuous function $f : A \longrightarrow \mathbf{R}$ is bounded. Prove the converse: If every continuous function $f : A \longrightarrow \mathbf{R}$ is bounded, then A is compact.

SOLUTION: We prove the contrapositive: if A is not compact, then there is a continuous unbounded function.

If A is not compact, then, since we are in $\mathbf{R}$, A is either not bounded or not closed. In the first case, the identity function $f(x) = x$ is continuous and unbounded.

If A is not closed, then a has a limit point $a \in \mathbf{R} - A$. Then the function

$$f(x) = \frac{1}{|x - a|}$$

is continuous and unbounded on A. ★

Note: This was your last problem session. Thank you all for your efforts!

Problem Session 11, Apr 17

Problem 1: Complete Metric Spaces. Let (X, d) be a metric space.

(a) If $A, B \subset X$ are subsets whose union is X and both A and B are complete with respect to d, show that X is also complete.
(b) Extend (a) to the case where X is the union of finitely many subsets $A_1, A_2, \ldots, A_n$.
(c) Show by example that (a) is not true for unions of infinitely many sets A_i.

Solution: We are going to use two facts about Cauchy sequences that we've seen before for sequences of real numbers: (1) If (x_n) is a Cauchy sequence in X, then every subsequence (x_{n_k}) is also a Cauchy sequence; (2) If a Cauchy sequence (x_n) has a convergent subsequence (x_{n_k}), then (x_n) converges to the same limit as (x_{n_k}). The proofs of (1) and (2) are provided below.

(a) Let now (x_n) be a Cauchy sequence in $X = A \cup B$. The either A or B contains infinitely many terms of (x_n). These terms form a subsequence (x_{n_k}) of (x_n) which is a Cauchy sequence by (1). Since this Cauchy sequence is in a complete metric space (A or B) it converges to a point of that space, which is also a point in X. Thus the Cauchy sequence (x_n) has a convergent subsequence and thus converges by (2).

(b) For $1 \le k \le n$, let $X_k = A_1 \cup A_2 \cup \ldots \cup A_k$. Thus $X_1 = A_1$ and $X_n = X$. We show by induction that X_k is complete for all k including $k = n$. The case where $k = 1$ is obvious. Suppose we already know that X_{k-1} is complete. Then $X_k = X_{k-1} \cup A_k$ and X_k is complete by (a). Thus $X = X_n$ is complete.

(c) Consider $X = \mathbf{Q}$ with the usual metric. The since $\mathbf{Q}$ is not closed in $\mathbf{R}$ it is not complete. On the other hand, $X = \cup_{x \in X} \{x\}$ and each $\{x\}$ is obviously complete.

We now prove (1) and (2). Let (x_n) be a Cauchy sequence in X and let (x_{n_k}) be subsequence of (x_n). Given $\varepsilon > 0$, choose $N \in \mathbf{N}$ such that $d(x_p, x_q) < \varepsilon$ whenever $p, q \ge N$. Then if $p, q \ge N$ we also have $n_p \ge p \ge N$ and $n_q \ge q \ge N$. Thus $d(x_{n_p}, x_{n_q}) < \varepsilon$. This shows (x_{n_k}) is Cauchy and proves (1).

For (2), suppose (x_n) is a Cauchy sequence and (x_{n_k}) is a convergent subsequence with limit x. Given $\varepsilon > 0$, let $N_1 \in \mathbf{N}$ be such that $d(x_{n_k}, x) < \varepsilon/2$ whenever $k \ge N_1$. Also, let $N_2 \in \mathbf{N}$ be such that $d(x_m, x_k) < \varepsilon/2$ whenever $m, k \ge N_2$. Let $N = \max\{N_1, N_2\}$ and take any $m \ge N$. Then $n_m \ge m \ge N$ and

$$d(x_m, x) \le d(x_m, x_{n_m}) + d(x_{n_m}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Thus (x_m) converges to x. ★

Problem 2: Connected Metric Spaces. A metric space (X, d) is *connected* if the only sets $A \subset X$ that are both open and closed are $\varnothing$ and X itself.

(a) State the intermediate value property for continuous functions $f : X \longrightarrow \mathbf{R}$, where (X, d) is a metric space.

(b) Show that (X, d) has the intermediate value property if and only if (X, d) is connected.

Solution: (a) A metric space (X, d) has the intermediate value property for continuous functions $f : X \longrightarrow \mathbf{R}$ if for every $a, b \in X$ and $r \in \mathbf{R}$ and every continuous $f : X \longrightarrow \mathbf{R}$ satisfying $f(a) < r < f(b)$, there is a $c \in X$ for which $f(c) = r$.

(b) Let (X, d) be connected, and let a, b, r and f be as above. We nee to show that there is a $c \in X$ such that $f(c) = r$. Suppose such c does not exist. Consider

$$U = \{x \in X \mid f(x) > r\}, \quad V = \{x \in X \mid f(x) < r\}.$$

Clearly $a \in V$ and $b \in U$ and we have $U \cup V = X$. This shows that V, U are both different from $\varnothing$ and from X. Moreover, since f is continuous, both U and V are open and therefore both are closed. This contradicts the assumption that X is connected.

Now suppose (X, d) is not connected and V is a set that is both open and closed, and different from $\varnothing$ and X. Define

$$f(x) = \begin{cases} 0 & \text{if } x \in V, \\ 1 & \text{if } x \in X - V. \end{cases}$$

Since the range of f is $\{0, 1\}$, f does not take on the value $1/2$. It remains to see that f is continuous. Let $W \subset \mathbf{R}$ be any open set. Then

$$f^{-1}(W) = \begin{cases} \varnothing & \text{if } 0 \notin W \text{ and } 1 \notin W, \\ V & \text{if } 0 \in W \text{ and } 1 \notin W, \\ U & \text{if } 0 \notin W \text{ and } 1 \in W, \\ X & \text{if } 0 \in W \text{ and } 1 \in W. \end{cases}$$

Thus $f^{-1}(W)$ is open in all cases and therefore f is continuous. ★

Problem 3: Uniform Convergence. State and prove the Continuous Limit Theorem (6.2.6) for functions $X \longrightarrow \mathbf{R}$, where (X, d) is a metric space.

SOLUTION: The Continuous Limit Theorem for functions $X \longrightarrow \mathbf{R}$ says that if a sequence (f_n) of continuous functions $f_n : X \to \mathbf{R}$ converges uniformly to $f : X \longrightarrow \mathbf{R}$, and each f_n is continuous at $x_0 \in X$ then f is also continuous at x_0.

Given $\varepsilon > 0$, there is an $N \in \mathbf{N}$ such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $n \geq N$ and $x \in X$. Since f_N is continuous at x_0, there is a $\delta > 0$ such that $|f_N(x) - f_N(x_0)| \leq \varepsilon/3$ whenever $d(x, x_0) < \delta$. Then if $d(x, x_0) < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &= \varepsilon. \end{aligned}$$

Thus f is continuous at x_0. ★

Problem Session 10, Apr 10

Here is a simple problem on the generalized (gauge) Riemann integral.

Problem : Modifying the Definition. Please review the definition of generalized Riemann-integrable functions and answer the following questions:

(a) What will happen if we in the definition of integrability we only consider *continuous* gauges?
(b) Same question if we only consider *monotone* gauges.

SOLUTION: (a) If $\delta : [a, b] \longrightarrow \mathbf{R}$ is a continuous gauge, then the is a constant $\delta_0 > 0$ satisfying $\delta(x) \geq \delta_0$. Since a δ_0-fine tagged partition is also a δ-fine tagged partition, we see that restricting ourselves to continuous gauges results in gauge-integrable function being Riemann-integrable.

(b) The same is true if we restrict ourselves to monotone gauges since if $\delta : [a, b] \longrightarrow \mathbf{R}$ is monotone, we again can find δ_0 satisfying $0 < \delta_0 \leq \delta(x)$ (for example, we can take $\delta_0 = \delta(a)$ if δ is increasing and $\delta_0 = \delta(b)$ if δ is decreasing). ★

Note: I gave up on Problem 2 — too much trouble writing down the solution.

Problem Session 9, Apr 3

Problem 7.4.5, modified. Let f and g be integrable functions on $[a, b]$.

(a) Show that if P is any partition of $[a, b]$, then

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

Provide a specific example where the inequality is strict. What does the corresponding inequality for lower sums look like?

(b) Prove that $f + g$ is integrable on $[a, b]$ and

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

SOLUTION: (a) Let P be any partition of $[a, b]$. For $x \in [x_{k-1}, x_k]$, we have

$$f(x) + g(x) \leq M_k(f) + M_k(g).$$

This implies

$$M_k(f + g) \leq M_k(f) + M_k(g),$$

and thererfore

$$U(f + g, P) \leq U(f, P) + U(g, P).$$

As an example, consider $f(x) = x$ and $g(x) = 1 - x$ on $[0, 1]$ and $P = \{0, 1/2, 1\}$. A quick computation shows that

$$U(f, P) = U(g, P) = 3/4, \quad U(f + g, P) = U(1, P) = 1 < U(f, P) + U(g, P).$$

A similar inequality, with a similar proof holds for lower sums:

$$L(f, P) + L(g, P) \leq L(f + g).$$

(b) Since f and g are integrable, given $\varepsilon > 0$, one can find two partitions P and Q of $[a, b]$ for which

$$U(f, P) - L(f, P) < \varepsilon/2 \quad \text{and} \quad U(g, Q) - L(g, Q) < \varepsilon/2.$$

Let $R = P \cup Q$ be a common refinement of P and Q. Then

$$U(f, R) - L(f, R) \leq U(f, P) - L(f, P) < \varepsilon/2,$$
$$U(g, R) - L(g, R) \leq U(g, Q) - L(g, Q) < \varepsilon/2,$$

and

$$\begin{aligned} U(f + g, R) - L(f + g, R) &\leq (U(f, R) + U(g, R)) - (L(f, R) + L(g, R)) \\ &= (U(f, R) - L(f, R)) + (U(g, R) - L(g, R)) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus $f + g$ is integrable on $[a, b]$. ★

Problem (Continuous Change of Variable). Let $f_n : [0, 1] \longrightarrow [0, 1]$ be the function $f_n(x) = x^n$, $n \geq 1$.

(a) Show that a set $A \subset [0,1]$ has measure zero if and only $f_n(A)$ has measure zero.
(b) Let $g : [0,1] \longrightarrow \mathbf{R}$ be a bounded function. Prove that $g \circ f_n : [0,1] \longrightarrow \mathbf{R}$ is integrable if and only if g is integrable.

SOLUTION: (a) We start with the following observation: a set $A \subset [0,1]$ has measure zero if and only if $A \cap [t,1)$ has measure zero for every $t \in (0,1)$. Since $A \cap [t,1)$ is a subset of A, if A has measure zero, then so does $A \cap [t,1)$. On the other hand, if each $A \cap [t,1)$ has measure zero, so does

$$\bigcup_{m \leq 1} A \cap [1/m, 1) = A \cap (0,1),$$

as a countable union of sets of measure zero, and then A has measure zero since $A - A \cap (0,1)$ has at most two points.

So let $A \subset [0,1]$ be a set of measure zero, and $B = f_n(A)$. Let $t \in (0,1)$ and $s = t^{1/n} \in (0,1)$. Then $A \cap [s,1)$ has measure zero and, given $\varepsilon > 0$, can be covered with countably many open intervals $O_k \subset (0,1)$ such that

$$\sum_k |O_k| < \varepsilon/n.$$

If $O_k = (\alpha, \beta)$, then $O_k' = (\alpha^n, \beta^n)$. From the Mean Value Theorem we see that

$$\beta^n - \alpha^n = nc^{n-1}(\beta - \alpha),$$

where $c \in (\alpha, \beta)$. Thus

$$|O_k'| = \beta^n - \alpha^n \leq n(\beta - \alpha) = n\,|O_k|$$

and

$$\sum_k |O_k'| \leq n \sum_k |O_k| < \varepsilon.$$

This proves that B has measure zero.

Now assume that $B \subset [0,1]$ has measure zero and $t \in (0,1)$. Then $B \cap [t,1)$ has measure zero. Thus, given $\varepsilon > 0$, we can cover B with countably many open intervals $O_k' \subset (0,1)$ such that

$$\sum_k |O_k'| < nt\varepsilon.$$

Then letting $O_k = f_n^{-1}(O_k')$, we see as above that for some $c \in O_k'$

$$|O_k| = (1/n)c^{(1/n)-1}\,|O_k'| < (1/n)t^{-1}\,|O_k'|$$

and

$$\sum_k |O_k| < (1/n)t^{-1} \sum_k |O_k'| < \varepsilon$$

Thus A has measure zero.

(b) Since both f_n and f_n^{-1} are continuous, $g \circ f_n$ is continuous at $c \in [0,1]$ if and only if g is continuous at $f_n(c)$. Thus the discontinuity set of $g \circ f_n$ has measure zero if and only if the discontinuity set of g has measure zero. Lebesgue's Theorem and (a) above now imply that $g \circ f_n$ is integrable if and only if g is integrable. ★

Problem Session 8, Mar 27

Problem 7.4.9, modified. Let g_n and g be uniformly bounded on $[0, 1]$, meaning that there exists a single $M > 0$ satisfying $|g(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \in N$ and $x \in [0, 1]$. Assume $(g_n) \to g$ uniformly on any interval $[0, \alpha]$, where $0 < \alpha < 1$.

If all g_n are integrable, show that g is integrable and $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

SOLUTION: To see that g is integrable, we note first that g is integrable on $[0, \alpha]$ for all $0 < \alpha < 1$. This readily follows from Theorem 7.4.4. Then, since g is bounded on $[0, 1]$, it is integrable on $[0, 1]$ by Theorem 7.3.2. It remains to see that $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 g$.

Let $\varepsilon > 0$. First choose α, such that $1 - \alpha < \varepsilon/4M$. Then, since $(g_n) \to g$ uniformly on $[0, \alpha]$, there exists an $N \in \mathbf{N}$ such that $|g(x) - g_n(x)| < \varepsilon/2$ for all $n \geq N$ and $x \in [0, \alpha]$. Thus

$$\begin{aligned}
\left|\int_0^1 g - \int_0^1 g_n\right| \leq \int_0^1 |g - g_n| &= \int_0^\alpha |g - g_n| + \int_\alpha^1 |g - g_n| \\
&< \varepsilon/2 + \int_\alpha^1 |g| + |g_n| \\
&\leq \varepsilon/2 + 2M(1 - \alpha) \\
&\leq \varepsilon/2 + \varepsilon/2 \\
&= \varepsilon.
\end{aligned}$$

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Problem 7.4.10, modified. Assume $g_n(x) = g(x^n)$ are integrable on $[0, 1]$ for $n \geq 1$ and that g is continuous at 0. Show

$$\lim_{n\to\infty} \int_0^1 g_n = g(0).$$

SOLUTION: We have $g_1 = g$. Thus g is integrable and therefore bounded on $[0, 1]$: For some $M > 0$ we have $|g(x)| \leq M$ for all $x \in [0, 1]$. This implies $|g_n(x)| = |g(x^n)| \leq M$ for all n and x. If h is the constant function $h(x) = g(0)$, we see that g_n and h are uniformly bounded by M.

Let $0 < \alpha < 1$. We now show that (g_n) converges uniformly to h on $[0, \alpha]$. Let $\varepsilon > 0$. Since $\lim_{x\to 0} g(x) = g(0)$, there is a $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ whenever $0 < x < \delta$. Since $(\alpha^n) \to 0$, there in an $N \in \mathbf{N}$ such that $\alpha^n < \delta$ for all $n \geq N$. Let $n \geq N$ and $x \in [0, \alpha]$. Then $x^n \leq \alpha^n < \delta$. Thus $|g_n(x) - h(x)| = |g(x^n) - g(0)| < \varepsilon$ and (g_n) converges to h uniformly on $[0, \alpha]$.

We can now apply problem 7.4.9 to conclude that

$$\int_0^1 g_n = \int_0^1 h = \int_0^1 g(0) = g(0).$$

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Problem Session 7, Mar 9

Problem 7.3.7. Assume $f : [a, b] \longrightarrow \mathbf{R}$ is integrable.

(a) Show that if $g(x) = f(x)$ for all but finitely many points in $[a, b]$, then g is integrable as well.

(b) Find an example to show that g may fail to be integrable if it differs from f at a countable number of points.

SOLUTION: (a) An obvious induction reduces the problem to the case where g differs from f at *one* point $c \in [a, b]$. If $c = a$ or $c = b$, g is integrable on $[a, b]$ by Theorem 7.3.2. Thus we will assume that $c \in (a, b)$.

Let $|f(c) - g(c)| = C > 0$. Given $\varepsilon > 0$, we can choose a partition P_1 for which

$$U(f, P_1) - L(f, P_1) < \frac{\varepsilon}{2}.$$

Then we can choose a refinement

$$P = \{x_0, x_1, \ldots, x_n\}$$

of P_1 for which $x_k - x_{k-1} < \varepsilon/8C$ for all $k = 1, 2, \ldots, n$. Then

$$m_k(f, P) - C \leq m_k(g, P) \leq M_k(g, P) \leq M_k(f, P) + C.$$

Thus

$$M_k(g, P) - m_k(g, P) \leq M_k(f, P) - m_k(f, P) + 2C$$

and, since c belongs to at most two intervals $[x_{k-1}, x_k]$,

$$U(g, P) - L(g, P) \leq U(f, P) - L(f, P) + 2(2C)\frac{\varepsilon}{8C} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(b) If g is the Dirichlet function

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}, \end{cases}$$

considered on any $[a, b]$ with $a < b$, and $f = 0$ on $[a, b]$, then f and g differ at countably many points of $[a, b]$, f is integrable, but g is not. ★

Problem Session 6, Feb 28

Problem 6.6.7. Find an example of each of the following or explain why no such function exists.

(a) An infinitely differentiable function $g(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $g(x)$ only for $x \in (-1, 1)$.

(b) An infinitely differentiable function $h(x)$ with the same Taylor series as $\sin(x)$ but such that $h(x) \neq \sin(x)$ for all $x \neq 0$.

(c) An infinitely differentiable function $f(x)$ on all of $\mathbf{R}$ with a Taylor series that converges to $f(x)$ if and only if $x \leq 0$.

SOLUTION: (a) Consider the series

$$\sum_{n=0}^{\infty} (-1)^n\, x^{2n}.$$

We know that this series converges for $|x| < 1$ to $g(x) = 1/(1 + x^2)$. Thus it is the Taylor series for $g(x)$. For $|x| \geq 1$, the series diverges since the the nth term $(-1)^n\, x^{2n}$ does not coverge to zero.

(b) Consider

$$h(x) = \begin{cases} \sin(x) + e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

That $h(x)$ has all the required properties immediately follows from the homework problem 6.6.6.

(c) Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Since all the derivatives of e^{-1/x^2} at 0 are 0, it is clear that $f(x)$ is an infinitely differentiable function on all of **R** and that its Taylor series at 0 is identically 0. Thus the Taylor series of $f(x)$ converges to $f(x)$ if and only if $x \leq 0$. ★

Problem 6.6.10 (a). Generate the Taylor series for

$$f(x) = \frac{1}{\sqrt{1-x}}$$

centered at zero, and use the Lagrange's Remainder Theorem to show that the series converges to f on $[0, 1/2]$. (The case $x < 1/2$ is more straightforward while $x = 1/2$ requires some extra care.) What happens when we attempt this with $x > 1/2$?

SOLUTION: It is easy to see (and prove by induction) that

$$f^{(n)}(x) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\left(\frac{7}{2}\right)\cdots\left(\frac{2n-1}{2}\right)(1-x)^{-(2n+1)/2}.$$

Thus the Taylor series for f centered at zero is

$$\sum_{n=0}^{\infty} a_n x^n,$$

where

$$a_n = \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\cdots\left(\frac{2n-1}{2n}\right).$$

The remainder $E_N(x)$ as given by Lagrange's theorem is

$$E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}\, x^{N+1} = a_{N+1}(1-c)^{-(2N+3)/2}\, x^{N+1} = a_{N+1}\left(\frac{x}{1-c}\right)^{N+1}(1-c)^{-1/2},$$

where $0 < |c| < |x|$. Note that $0 < a_{N+1} < 1$ and that if $|x| < 1/2$ we have $|c| < 1/2$ and thus $1 - c > 1/2$. It follows that if we fix any α satisfying $0 < \alpha < 1/2$, we have

$$|E_N(x)| < \alpha^{N+1}(1-c)^{-1/2} < \alpha^{N+1}\sqrt{2}$$

for all $x \in [-\alpha, \alpha]$. This shows that the Taylor series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly to f on $(-1/2, 1/2)$.

For $x = 1/2$, we have

$$|E_N(x)| < a_{N+1}\sqrt{2}$$

and it will be sufficient to show that $\lim_{n\to\infty} a_n = 0$. We have

$$\frac{1}{a_n} = \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right)\left(1 + \frac{1}{7}\right)\cdots\left(1 + \frac{1}{2n-1}\right)$$

and $\lim_{n\to\infty} 1/a_n = 0$ since the series

$$\sum_{n=1}^{\infty} \frac{1}{2n-1}$$

diverges (see problem 2.4.10). Thus $\lim_{n\to\infty} a_n = 0$ and we conclude that the Taylor series for

$$f(x) = \frac{1}{\sqrt{1-x}}$$

converges to $f(x)$ at $x = 1/2$ and therefore converges uniformly to f on $[0, 1/2]$ by Abel's Theorem 6.5.4. ★

Problem Session 5, Feb 21

Problem 6.4.2. Decide whether each proposition is true or false, providing short justification or counterexample as appropriate.

(a) If $\sum_{n=1}^{\infty} g_n$ converges uniformly, then (g_n) converges uniformly to zero.
(b) If $0 \le f_n(x) \le g_n(x)$ and $\sum_{n=1}^{\infty} g_n$ converges uniformly, then $\sum_{n=1}^{\infty} f_n$ converges uniformly.
(c) If $\sum_{n=1}^{\infty} f_n$ converges uniformly on A, then there exist constants M_n such that $|f_n(x)| \le M_n$ for all $x \in A$ and $\sum_{n=1}^{\infty} M_n$ converges.

Solution: (a) **True.** Let $\varepsilon > 0$ be any number. The Cauchy criterion guarantees the existence of a $N \in \mathbf{N}$ such that

$$|g_n(x) + g_{n+1}(x) + \cdots + g_{m-1}(x)| < \varepsilon$$

for all x and all $m > n \ge N$. This is true, in particular for all $n \ge N$ and $m = n+1$, i.e. $|g_n(x)| < \varepsilon$. Thus (g_n) converges uniformly to zero.

(b) **True.** Let $\varepsilon > 0$ be any number. The Cauchy criterion guarantees the existence of a $N \in \mathbf{N}$ such that

$$|g_n(x) + g_{n+1}(x) + \cdots + g_{m-1}(x)| < \varepsilon$$

for all x and all $m > n \ge N$. Since $0 \le f_n(x) \le g_n(x)$ for all n and x, we have

$$\begin{aligned} |f_n(x) + f_{n+1}(x) + \cdots + f_{m-1}(x)| &= f_n(x) + f_{n+1}(x) + \cdots + f_{m-1}(x) \\ &\le g_n(x) + g_{n+1}(x) + \cdots + g_{m-1}(x) < \varepsilon. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} f_n$ converges uniformly by the Cauchy criterion.

(c) **False.** Consider the constant functions $f_n(x) = (-1)^n/n$. As is well-known, $\sum_{n=1}^{\infty} f_n$ converges conditionally. Since $|f_n(x)| \leq M_n$ implies $M_n \geq 1/n$, we see that $\sum_{n=1}^{\infty} M_n$ diverges. ★

Problem 6.5.2. Find suitable coefficients (a_n) so that the resulting power series $\sum a_n x^n$ has the given properties, or explain why such a request is impossible.

(a) Converges for every value of $x \in \mathbf{R}$.
(b) Diverges for every value of $x \in \mathbf{R}$.
(c) Converges absolutely for all $x \in [-1, 1]$ and diverges off of this set.
(d) Converges conditionally at $x = -1$ and converges absolutely at $x = 1$.
(e) Converges conditionally at both $x = -1$ and $x = 1$.

Solution: (a) $a_n = 0$ for all $n \geq 0$.

(b) Impossible: All power series converge at 0.

(c) Let $a_0 = 0$ and $a_n = 1/n^2$ for $n \geq 1$. To see that $\sum a_n x^n$ converges absolutely at $x = \pm 1$ it is enough to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. This follows readily from

$$\frac{1}{n^2} < \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1.$$

(d) Since $\sum |a_n(-1)^n| = \sum |a_n(1)^n|$, this request is impossible.

(e) Let

$$a_{2n} = 0 \quad \text{and} \quad a_{2n+1} = \frac{1}{2n+1} \qquad (n \geq 0).$$

The two series $f(x) = \sum a_n x^n$ is the Taylor series of $f(x) = \arctan(x)$ (discussed in class on February 17). The series $f(1)$ and $f(-1)$ are both alternating and thus converge. On the other hand,

$$\sum |a_n| = \sum \frac{1}{2n+1}$$

diverges, which can be seen from

$$\frac{1}{2n+1} > \frac{1}{2n+2}$$

and the fact that the harmonic series

$$\sum_{n=0}^{\infty} \frac{1}{2n+2} = (1/2) \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Problem Session 4, Feb 14

Problem 6.3.2. Consider the sequence of functions

$$h_n(x) = \sqrt{x^2 + \frac{1}{n}}.$$

(a) Compute the pointwise limit of (h_n) and then prove that the convergence is uniform on $\mathbf{R}$.
(b) Note that each h_n is differentiable. Show that $g(x) = \lim h_n'(x)$ exists for all x, and explain how we can be certain that the convergence is *not* uniform on any neighborhood of zero.

SOLUTION: (a) Clearly, $\lim h_n(x) = \sqrt{x^2} = |x|$. Moreover

$$\sqrt{x^2 + 1/n} - \sqrt{x^2} = \frac{1/n}{\sqrt{x^2 + 1/n} + \sqrt{x^2}} \leq \frac{1}{\sqrt{n}}.$$

Since $\lim \dfrac{1}{\sqrt{n}} = 0$, we see that (h_n) converges uniformly.

(b) We have

$$h_n'(x) = \frac{x}{\sqrt{x^2 + 1/n}}$$

for all x. If $g(x) = \lim h_n'(x)$, we have $g(0) = 0$ and if $x \neq 0$,

$$g(x) = \lim h_n'(x) = \frac{x}{\sqrt{x^2}} = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0. \end{cases}$$

Since $\lim h_n$ is not differentiable on any interval containing zero, (h_n') could not converge uniformly on any such interval, since this would violate Theorem 6.3.3. ★

Problem 6.3.6. Provide an example or explain why the request is impossible. Let's take the domain of the functions to be all of $\mathbf{R}$.

(a) A sequence (f_n) of nowhere differentiable functions with $(f_n) \to f$ uniformly and f everywhere differentiable.
(b) A sequence (f_n) of differentiable functions such that (f_n') converges uniformly but the original sequence (f_n) does not converge for any $x \in \mathbf{R}$.
(c) A sequence (f_n) of differentiable functions such that both (f_n) and (f_n') converge uniformly, but $f = \lim f_n$ is not differentiable at some point.

SOLUTION: (a) Let g be the Dirichlet function (see sectioin 4.1) and let $f_n(x) = g(x)/n$. Thus

$$f_n(x) = \begin{cases} 1/n, & \text{if } x \in \mathbf{Q}, \\ 0, & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Since g is nowhere differentiable, the same is true for f_n. On the other hand, (f_n) clearly converges uniformly to the constant function $f = 0$, which is everywhere differentiable.

(b) Let f be *any* differentiable function on $\mathbf{R}$ and $f_n(x) = f(x) + n$. Then $f_n'(x) = f'(x)$ and therefore (f_n') converges to f, uniformly. On the other hand $(f_n(x))$ does not converge for any x.

(c) This is not possible, since it would violate Theorem 6.3.3. ★

Problem Session 3, Feb 7

Problem 5.2.9. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational values.
(b) If f' exists on an open interval and there is some point c where $f'(c) > 0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(c) > 0$ for all $x \in V_\delta(c)$.
(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0} f'(x) = L$, then it must be that $L = f'(0)$.

Solution: (a) **True.** If for some a, b in the interval we have $f'(a) \neq f'(b)$, then Darboux's theorem implies that $f'(x)$ takes on all the values in the open interval with endpoints $f'(a)$ and $f'(b)$, in particular, uncountably many irrational values.

(b) **False.** Consider the function

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

We have

$$f'(0) = \lim_{x\to 0} (1 + 2x \sin(1/x)) = 1 > 0.$$

However, for $x \neq 0$,

$$f'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x),$$

and if we let $x_k = 1/2\pi k$, we have $\lim (x_k) = 0$ and $f'(x_k) = -1$. Thus every δ-neighborhood $V_\delta(0)$ contains points x satisfying $f'(x) < 0$.

(c) **True.** Let's assume, for simplicity, that f is differentiable on the interval $[a, b]$ containing 0 and $b > 0$. The case where $b = 0$, i.e. $a < 0$ is completely similar. Replacing $f(x)$ with $f(x) - Lx$, we can also assume that $\lim_{x\to 0} f'(x) = 0$. We need to show that $f'(0) = 0$. So suppose $f'(0) \neq 0$ and let $\varepsilon = |f'(0)|/2 > 0$. Then there is a $\delta > 0$ such that $[0, \delta] \subset [a, b]$ and $|f'(x)| < \varepsilon$ for all $x \in (0, \delta)$. This means, in particular, that $f'(x) \neq 3f'(0)/4$ in $(0, \delta)$ thus contradicting Darboux's theorem on $[0, \delta]$. ★

Problem Session 2, Jan 31

Problem 5.2.6 (b). Let g be defined on an open interval A and $c \in A$. If g is differentiable at $c \in A$, show

$$g'(c) = \lim_{h\to 0} \frac{g(c+h) - g(c-h)}{2h}.$$

SOLUTION: We have

$$\begin{aligned}\lim_{h\to 0}\frac{g(c+h)-g(c-h)}{h} &= \lim_{h\to 0}\left(\frac{g(c+h)-g(c)}{h}+\frac{g(c-h)-g(c)}{-h}\right)\\ &= \lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{g(c-h)-g(c)}{-h}\\ &= \lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}\\ &= 2g'(c).\end{aligned}$$

★

Problem 5.3.12. If f is twice differentiable on an open interval containing a and f'' is continuous at a, show

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(a).$$

SOLUTION: Let

$$g(h)=f(a+h)-2f(a)+f(a-h).$$

This function of h is defined on an open interval containing 0 and we have

$$g'(h)=f'(a+h)-f'(a-h)$$

and

$$g''(h)=f''(a+h)+f''(a-h).$$

Thus, since f'' is continuous at a,

$$\lim_{h\to 0}\frac{f''(a+h)+f''(a-h)}{2}=f''(a).$$

On the other hand, $g'(0)=0$ and L'Hospital's Rule implies

$$\lim_{h\to 0}\frac{f'(a+h)-f'(a-h)}{2h}=f''(a).$$

Similarly, since $g(0)=0$, the same L'Hospital's Rule implies

$$\lim_{h\to 0}\frac{f(a+h)-2f(a)+f(a-h)}{h^2}=f''(a).$$

★

Problem Session 1, Jan 24

Problem 5.2.9.. Decide whether each conjecture is true or false. Provide an argument for those that are true and a counterexample for each one that is false.

(a) If f' exists on an interval and is not constant, then f' must take on some irrational value.
(b) If f' exists on an interval and there is some point c where $f'(c)>0$, then there exists a δ-neighborhood $V_\delta(c)$ around c in which $f'(x)>0$ for all $x\in V_\delta(c)$.
(c) If f is differentiable on an interval containing zero and if $\lim_{x\to 0}f'(x)=L$, then it must be that $L=f'(0)$.

Solution: (a) **True.** Suppose f' exists and is not constant on an interval A. Then there are $a < b$ in A such that $f'(a) \neq f'(b)$. The density of irrational numbers in $\mathbf{R}$ implies that there is an irrational α satisfying $f'(a) < \alpha < f'(b)$ or $f'(b) < \alpha < f'(a)$. Darboux's Theorem now implies that $f'(c) = \alpha$ for some c between a and b. Thus f' takes on an irrational value.

(b) **False.** Consider the function

$$f(x) = \begin{cases} 2x^2 \sin(1/x) + x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

By a direct computation, we see that for $x \neq 0$, we have

$$f'(x) = 4x \sin(1/x) - 2\cos(1/x) + 1,$$

and

$$f'(0) = \lim_{x\to 0} \frac{2x^2 \sin(1/x) + x}{x} = \lim_{x\to 0}\ 2x\sin(1/x) + 1 = 1.$$

In particular, $f'(0) > 0$. Consider any $\delta > 0$, and choose $k \in \mathbf{N}$ so that

$$\frac{1}{2k\pi} < \delta.$$

If

$$x = \frac{1}{2k\pi},$$

we have $x \in V_\delta(0)$ and

$$f'(x) = \frac{2}{k\pi}\sin(2k\pi) - 2\cos(2k\pi) + 1 = -1.$$

This shows that there is no $\delta > 0$ such that $f'(x) > 0$ for all $x \in V_\delta(0)$.

(c) **True.** Suppose f' is defined on an interval A. Without loss of generality, we will assume that 0 is not the right end-point of A. (The case when 0 is not the left end-point is completely similar and clearly 0 cannot be both the right and left end-point of A.) Let's assume (seeking contradiction) that $L \neq f'(0)$, and let

$$\varepsilon = \frac{|L - f'(0)|}{2}.$$

Since $\lim_{x\to 0} f'(x) = L$, there is a $\delta > 0$ such that for every x satisfying $0 < x < \delta$, we have $|f'(x) - L| < \varepsilon$. This is true, in particular for $x = \delta/2$. We will further assume that δ is small enough to be in A. We now show that f' does not have the Intermediate Value Property on $[0, \delta/2]$ and thus contradicts Darboux's theorem.

If $L < f'(0)$, choose an α satisfying $f'(0) - \varepsilon < \alpha < f'(0)$. If $L > f'(0)$, choose an α satisfying $f'(0) < \alpha < f'(0) + \varepsilon$. In both cases, α is between $f'(0)$ and $f'(\delta/2)$ and $|\alpha - L| > \varepsilon$. Therefore, $\alpha \neq f'(c)$ for any $c \in (0, \delta/2)$, contradicting Darboux's theorem. ★

Notes: A. The proof in part (c) uses only one property of f', the Intermediate Value Property (Darboux's Theorem). In other words, the following more general statement is true:

Let A be an interval and let $g : A \longrightarrow \mathbf{R}$ be any function having the Intermediate Value Property (i.e. for every $a < b$ with $a, b \in A$, if $g(a) < \alpha < g(b)$ or $g(a) > \alpha > g(b)$, *then $g(c) = \alpha$ for some $c \in (a,b)$.) Then g cannot have any removable discontinuities in A. In other words, if for some $p \in A$ we have $\lim_{x\to p} g(x) = L$, then $L = g(p)$.*

B. Another, stronger, version of the same result is the subject of Exercise 5.3.8. It says:

Assume that f is continuous on an interval containing zero and differentiable for all $x \neq 0$. If $\lim_{x\to} f'(x) = L$, then $f'(0)$ exists and equals L.

Proof. Given any $\varepsilon > 0$, find a $\delta > 0$ such that if $x \in A$ and $0 < |x| < \delta$, then $|f'(x) - L| < \varepsilon$.

Take any $x \in A$ such that $0 < x < \delta$, and apply the Mean Value Theorem to f on $[0,x]$: there is a $c \in (0,x)$ such that

$$\frac{f(x) - f(0)}{x} = f'(c).$$

Then

$$\left|\frac{f(x) - f(0)}{x} - L\right| = |f'(c) - L| < \varepsilon,$$

since $|c| = c < x < \delta$. This shows that

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x}$$

exists and equals L.

Similarly, choosing $x \in A$ such that $-\delta < x < 0$, and applying the Mean Value Theorem to f on $[x,0]$, we see that

$$\lim_{x\to 0^-} \frac{f(x) - f(0)}{x}$$

exists and equals L. Thus

$$\lim_{x\to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x} = L,$$

i.e. $f'(0) = L$. ★
