In this note we prove a version of the Lebesgue Number Lemma and apply it to prove that the Cantor-like set D discussed in the solutions to Exam 2 does not have measure zero. The Lebesgue Number Lemma holds in any metric space and its proof is not any more complicated in the metric space setting.

Let $X, d)$ be a metric space and let $A \subset X$ be any set. For $x \in X$, we define the distance $d(x, A)$ by

$$d(x, A) = \inf \{d(x, a) \mid a \in A\}.$$

Lemma 1. *The function $f(x) = d(x, A)$ is a continuous function $f : X \longrightarrow \mathbf{R}$.*

Proof. Let $x, y \in X$. Then for $a \in A$ we have

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

Thus

$$d(x, A) - d(x, y) \leq d(y, a)$$

and since this holds for all $a \in A$, we have

$$d(x, A) - d(x, y) \leq d(y, A).$$

Thus

$$d(x, A) - d(y, A) \leq d(x, y).$$

Interchanging the roles of x and y, we get

$$d(y, A) - d(x, A) \leq d(x, y)$$

and therefore

$$|d(x, A) - d(y, A)| \leq d(x, y).$$

This proves the continuity and even uniform continuity of f. ★

Lemma 2. *If A is closed in X, $d(x, A) = 0$ if and only if $x \in A$.*

Proof. If $x \in A$, take $a = x \in A$. Then

$$0 \leq d(x, A) \leq d(x, a) = 0.$$

Thus $d(x, A) = 0$.

Conversely, if $d(x, A) = 0$, then for every $n \in \mathbf{N}$ there is an $a_n \in A$ such that $d(x, a_n) < 1/n$. It follows that the sequence (a_n) converges to x and since A is closed, $x \in A$. ★

For a bounded set $S \subset X$, we define the diameter

$$\text{diam}(S) = \sup \{d(a, b) \mid a, b \in S\}.$$

Theorem 1. *Let $K \subset X$ be a compact set and $\mathcal{U}$ be any covering of K with open sets in X. Then there is a number $\delta > 0$ such that for every set $S \subset X$ such that $K \cap S \neq \varnothing$, $S \subset U$ for some $U \in \mathcal{U}$.*

The number δ is referred to as a *Lebesgue number* of the covering $\mathcal{U}$. The following proof is adapted from the proof given in *Topology*, by Mankres.

Proof. Since K is compact, we can replace $\mathcal{U}$ by a finite subcover

$$\{U_1, U_2, \ldots U_n\}.$$

Let $A_i = X - U_i$. These are closed subsets of X and $\cap A_i$ does not intersect K. Define a function $g : K \longrightarrow \mathbf{R}$ by

$$g(x) = \frac{1}{n}\sum_{i=1}^{n} d(x, A_i).$$

By Lemma 1, g is a continuous function on K and since $K \cap A_1 \cap A_2 \cap \ldots \cap A_n = \varnothing$, $x \notin A_i$ for at least one index i and then Lemma 2 implies that $g(x) > 0$.

By the extreme value theorem, g has a minimum $\delta > 0$. We now show that δ has the required property. Let $S \subset X$ be any set satisfying

$$K \cap S \neq \varnothing, \quad \operatorname{diam}(S) < \delta.$$

Choose any point $a \in K \cap S$. Then

$$\delta \leq g(a) \leq d(a, A_m),$$

where $d(a, A_m)$ is the largest of the numbers $d(a, A_i)$ (recall that $g(a)$ is the *average* of the numbers $d(a, A_i)$. For every $b \in S$, we have

$$d(a, b) < \delta \leq d(a, A_m).$$

Thus $b \notin A_m$, i.e. $b \in U_m$. Since this holds for all $b \in S$, we conclude that $S \subset U_m$. ★

We now use Theorem 1 to prove that the Cantor-like set D described in the solution to Question 5 of Exam 2 does not have measure zero. Please refer to the posted solution for the construction of D.

Theorem 2. *The set D does not have measure zero.*

Proof. Let $\mathcal{J}$ be any covering of D with countably many open intervals in $\mathbf{R}$. We show that the total length of these intervals is $> 2/3$. Since D is compact, we may assume, without loss of generality that $\mathcal{J}$ is finite, i.e.

$$\mathcal{J} = \{J_1, J_2, \ldots, J_m\}.$$

Let $\delta > 0$ be a Lebesgue number for $\mathcal{J}$. Choose $n \geq 1$ such that $1/2^n < \delta$. Recall that D_n is a disjoint union of 2^n closed intervals if total length $> 2/3$. The diameter of each of these intervals is $< 1/2^n < \delta$ and it is easy to see from the construction of D that the endpoints of these 2^n intervals are in D. Thus each of the 2^n intervals intersects D and therefore is contained in some J_i. We now see that D_n is contained in the union of the J_i and thus the sum of the lengths of the J_i is $> 2/3$. ★