Math 213 - Limits and Continuity

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Homework

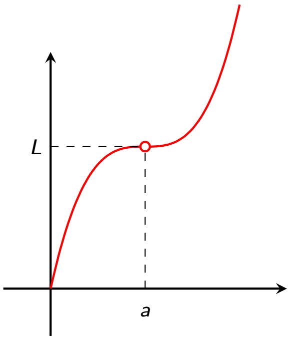
- Re-read section 14.2
- Start working on practice problems in section 14.2, 1, 5-17 (odd), 21, 25, 29, 31, 33, 35
- Be ready to work on sections 14.1-14.2 in recitation tomorrow
- Read section 14.3 for Wednesday's lecture

Unit II: Differential Calculus of Several Variables

Goals of the Day

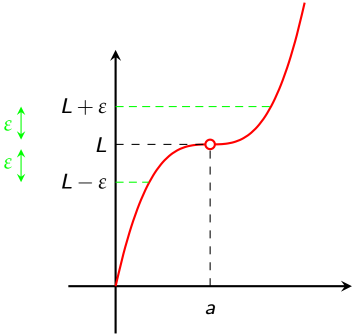
- Understand limits for functions of two variables and know how to determine when they do or do not exist
- Understand what a continuous function of two variables is and how to determine when a given such function is continuous
- Understand how limits and continuity generalize to functions of three variables

Limits: One Variable



$\lim_{x \to a} f(x) = L$ if:

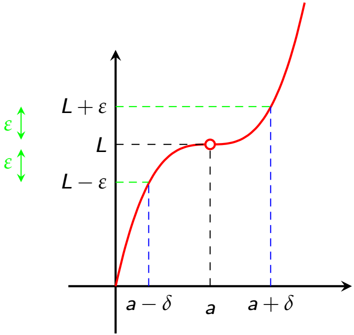
Limits: One Variable



$\lim_{x\to a} f(x) = L$ if:

- for every $\varepsilon > 0$,

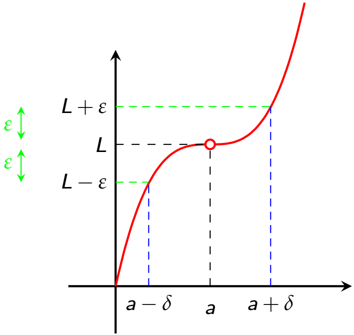
Limits: One Variable



$\lim_{x\to a} f(x) = L$ if:

- for every $\varepsilon > 0$,
- we can find a $\delta > 0$ so that, if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$

Limits: One Variable



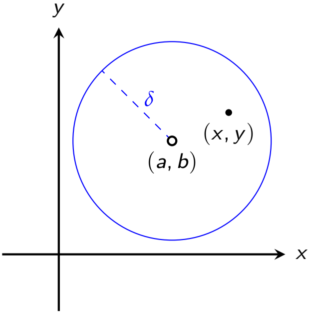
$\lim_{x\to a} f(x) = L$ if:

- for every $\varepsilon > 0$,
- we can find a $\delta > 0$ so that,
  if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$

Remember that f does not need to be defined at $x = a$ for the limit to exist

One can approach $x = a$ either from the left ($x < a$) or from the right ($x > a$)

Limits: Two Variables



$\lim_{(x,y)\to(a,b)} f(x,y) = L$ if:

- for any $\varepsilon > 0$,
- we can find a $\delta > 0$ so that that, if

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

then $|f(x,y) - L| < \varepsilon$

f does not need to be defined at (a,b)

One can approach $(x,y) = (a,b)$ on *any* line (or *any curve*!) that goes to (a,b)

Limits - Two Variables

The set

$$D = \left\{ (x, y) : 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \right\}$$

is a *punctured disk* of radius δ at (a, b).

$\lim_{(x,y)\to(a,b)} f(x, y) = L$ if:

given any $\varepsilon > 0$, we can a $\delta > 0$ so that

for any (x, y) in D, $f(x, y)$ is within ε of L

When don't limits exist?

Suppose

$$f(x,y) = \frac{x^2y}{x^3+y^3}$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Try the *line test*: $(x,y) = (x,mx)$

$$f(x,mx) = \frac{mx^3}{x^3+m^3x^3} = \frac{m}{1+m^3}$$

What does this tell you about the limit?

Now You Try It

Find the limit or show that the limit does not exist.

1. $\lim_{(x,y)\to(3,2)} (x^2y^3)$
2. $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x-y)$
3. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$
4. $\lim_{(x,y)\to(0,0)} \dfrac{xy^4}{x^4+y^4}$
5. $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$
6. $\lim_{(x,y)\to(0,0)} \dfrac{e^{-x^2-y^2}-1}{x^2+y^2}$

Post-Lecture Solutions, I

1. $\lim_{(x,y)\to(3,2)} x^2y^3 = 3^2 2^3 = 72$
2. $\lim_{(x,y)\to(\pi,\pi/2)} y\sin(x-y) = \pi\sin(\pi/2) = \pi$
3. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{r^2\cos\theta\sin\theta}{r}\right| \leq r$$

so

$$0 \leq |f(x,y)| \leq r.$$

By the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$

Post-Lecture Solutions, II

4. Here's a different solution from the one given in class. We can estimate

$$\left|\frac{xy^4}{x^4+y^4}\right| \leq |x| \left|\frac{x^4+y^4}{x^4+y^4}\right| \leq |x|$$

so

$$0 \leq \left|\frac{xy^4}{x^4+y^4}\right| \leq |x|$$

so by the Squeeze Theorem again, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

5. For this one use the line test. Compute that

$$f(x, mx) = \frac{x(mx)}{x^2+(mx)^2} = \frac{m}{1+m^2}$$

which will given different limits depending on the value of m. So this function has *no* limit as $(x, y,) \to (0, 0)$.

Post-Lecture Solutions, III

6. Using polar coordinates

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} \frac{e^{-r^2}-1}{r^2}$$

$$= \lim_{r\to 0} \frac{-2re^{-r^2}}{2r} = -1$$

Continuity

One Variable

A function $f(x)$ is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a)$$

Recall this means:

- a lies in the domain of f
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

Two Variables

A function $f(x, y)$ is continuous at a point (a, b) of its domain if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

Deduce that this means:

- (a, b) lies in the domain of f
- $\lim_{(x,y) \to (a,b)} f(x, y)$ exists
- $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$

Now You Try It

Determine the set of points at which each function is continuous.

1. $f(x,y) = \dfrac{xy}{1+e^{x-y}}$
2. $f(x,y) = \dfrac{1+x^2+y^2}{1-x^2-y^2}$
3. $g(x,y) = \ln(1+x-y)$
4. $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$, $(x,y) \neq (0,0)$
   $f(0,0) = 0$

Post-Lecture solutions:

1. Continuous for all (x,y) in $\mathbb{R}^2$.
2. Continuous for all (x,y) with $x^2+y^2 \neq 1$
3. Continuous for all (x,y) with $1+x-y > 0$, i.e., $1+x > y$
4. Continuous for all (x,y) in $\mathbb{R}^2$. One has to check that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$. You can do this either using polar coordinates or using the fact that $\frac{x^2y^2}{x^2+y^2} \leq x^2 \frac{y^2}{x^2+y^2} \leq x^2$ and using the Squeeze Theorem.

Three Variables

Let $\mathbf{x} = (x, y, z)$, $\mathbf{a} = (a, b, c)$

$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ if for every $\varepsilon > 0$, there is a $\delta > 0$ so that if $\mathbf{x}$ is in the domain of f and

$$0 < |\mathbf{x} - \mathbf{a}| < \delta,$$

then

$$|f(\mathbf{x}) - L| < \varepsilon$$

$f(\mathbf{x})$ is continuous at a point $\mathbf{a}$ in its domain if

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

Three Variables

1. Describe the set of points at which the function

$$f(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2}$$

is continuous

2. Do the same for

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$