Math 213 - Applications of Double Integrals

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October 26, 2018

Homework

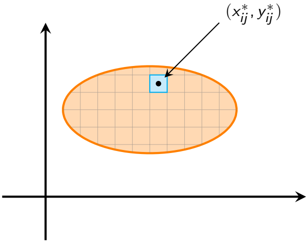
- Re-read section 15.4
- Begin work on problems 1-23 (odd) from 15.4
- Read section 15.5 for Monday, October 29
- Finish up Webwork C1

Unit III: Multiple Integrals

Goals of the Day

- Compute Mass of a Lamina from Mass Density
- Compute Moments and Centers of Mass
- Compute Moments of Inertia

Density to Mass



The *density* of a plane solid (lamina) D is the mass per unit area:

$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}.$$

The total mass is approximately

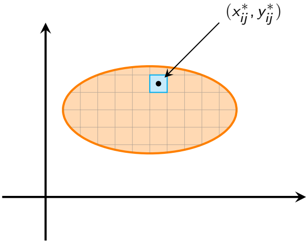
$$M \simeq \sum_{i,j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A$$

where (i, j) index rectangles in D of area ΔA

The total mass is *exactly*

$$M = \iint_D \rho(x, y)\, dA$$

Charge Density to Charge



The *charge density* of a plane conductor D is the mass per unit area:

$$\rho(x, y) = \lim_{\Delta A \to 0} \frac{\Delta Q}{\Delta A}.$$

The total mass is approximately

$$Q \simeq \sum_{i,j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A$$

where (i, j) index rectangles in D of area ΔA

The total mass is *exactly*

$$Q = \iint_D \rho(x, y)\, dA$$

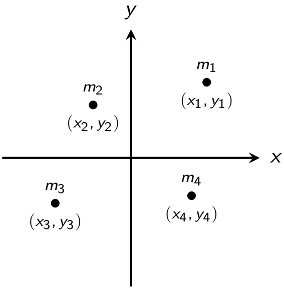
Density to Mass

Electric charge is distributed over the disc $x^2 + y^2 \leq 1\text{m}$ with charge density

$$\sigma(x, y) = \sqrt{x^2 + y^2}$$

coulombs per square meter. Find the total charge Q on the disc.

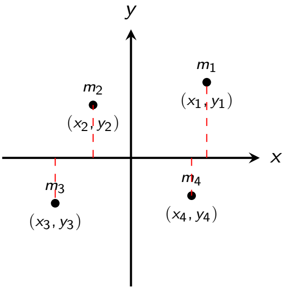
Moments, Centers of Mass - Discrete Systems



For a set of point masses m_i at (x_i, y_i):

- The total mass is $M = \sum_{i=1}^{n} m_i$

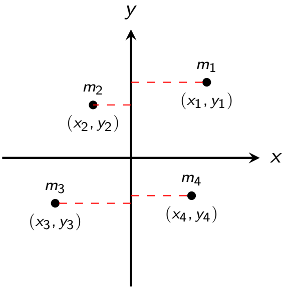
Moments, Centers of Mass - Discrete Systems



For a set of point masses m_i at (x_i, y_i):

- The total mass is $M = \sum_{i=1}^{n} m_i$
- The moment with respect to the x-axis is $M_x = \sum_{i=1}^{n} y_i m_i$

Moments, Centers of Mass - Discrete Systems



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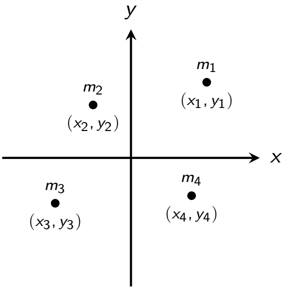
- The moment with respect to the y-axis is $M_y = \sum_{i=1}^{n} x_i m_i$

Moments, Centers of Mass - Discrete Systems



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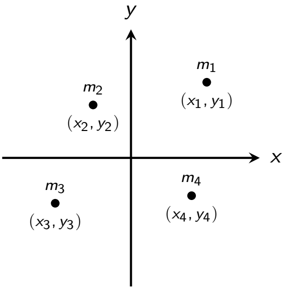
- The moment with respect to the y-axis is $M_y = \sum_{i=1}^{n} x_i m_i$
- The center of mass is

$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

Moments, Centers of Mass - Discrete Systems



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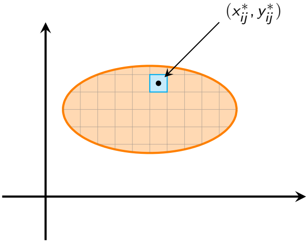
$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$$

$$(\overline{x}, \overline{y}) = \left(\frac{\sum_{i=1}^{n} x_i m_i}{\sum_{i=1}^{n} m_i}, \frac{\sum_{i=1}^{n} y_i m_i}{\sum_{i=1}^{n} m_i} \right)$$

Moments, Centers of Mass - Continuous Systems



If a lamina D has density $\rho(x, y)$:

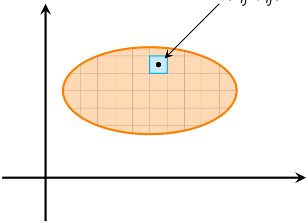
$$M \simeq \sum_{ij} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

$$M_x \simeq \sum_{ij} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

$$M_y \simeq \sum_{ij} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

Taking limits. . .

Moments, Centers of Mass - Continuous Systems



If a lamina D has density $\rho(x, y)$:

$$M \simeq \sum_{ij} \rho(x^*_{ij}, y^*_{ij}) \Delta A$$

$$M_x \simeq \sum_{ij} y^*_{ij} \rho(x^*_{ij}, y^*_{ij}) \Delta A$$

$$M_y \simeq \sum_{ij} x^*_{ij} \rho(x^*_{ij}, y^*_{ij}) \Delta A$$

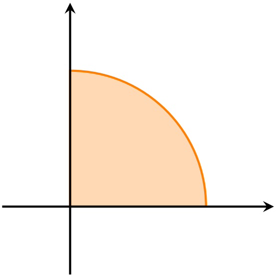
Taking limits. . .

$$M = \iint_D \rho(x, y)\, dA$$

$$M_x = \iint_D y\rho(x, y)\, dA$$

$$M_y = \iint_D x\rho(x, y)\, dA$$

Center of Mass



$$M \simeq \sum_{ij} \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

$$M_x \simeq \sum_{ij} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

$$M_y \simeq \sum_{ij} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A$$

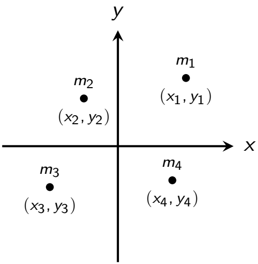
A lamina occupies the region

$$D = \{(x, y) : x^2 + y^2 \leq 1, \ x \geq 0, \ y \geq 0\}.$$

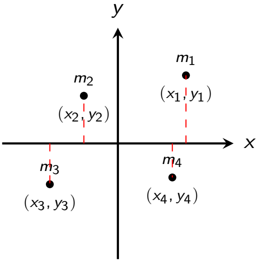
Find its center of mass if the density at each point is proportional to its distance from the x-axis.

Moment of Inertia - Discrete Systems

For a set of point masses m_i at (x_i, y_i):



Moment of Inertia - Discrete Systems

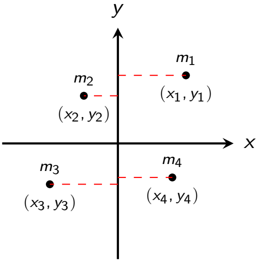


For a set of point masses m_i at (x_i, y_i):

- The *moment of inertia about the x-axis* is

$$I_x = \sum_{i=1}^{n} m_i y_i^2$$

Moment of Inertia - Discrete Systems



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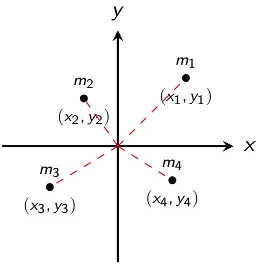
- The *moment of inertia about the y-axis* is

$$I_y = \sum_{i=1}^{n} m_i x_i^2$$

Moment of Inertia - Discrete Systems



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- The *moment of inertia about the x-axis* is

$$I_x = \sum_{i=1}^{n} m_i y_i^2$$

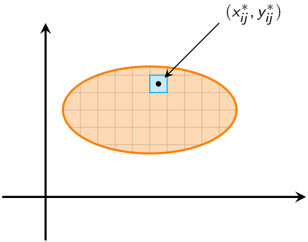
- The *moment of inertia about the y-axis* is

$$I_y = \sum_{i=1}^{n} m_i x_i^2$$

- The *moment of intertia about the origin* is

$$I_0 = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)$$

Moment of Inertia - Continuous Systems



If a lamina D has density $\rho(x, y)$:

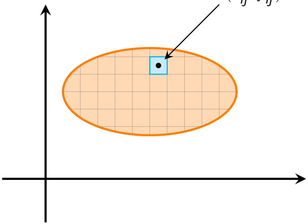
$$I_x \simeq \sum_{ij} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

$$I_y \simeq \sum_{ij} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

$$M_y \simeq \sum_{ij} \left((x_{ij}^*)^2 + (y_{ij}^*)^2 \right) \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

Taking limits. . .

Moment of Inertia - Continuous Systems



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$$M_y \simeq \sum_{ij} \left((x_{ij}^*)^2 + (y_{ij}^*)^2 \right) \rho(x_{ij}^*, y_{ij}^*) \Delta A$$

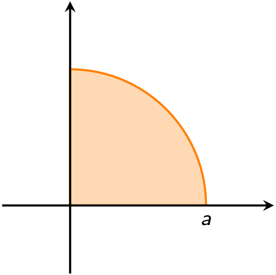
Taking limits...

$$I_x = \iint_D y^2 \rho(x, y)\, dA$$

$$I_y = \iint_D x^2 \rho(x, y)\, dA$$

$$I_0 = \iint_D (x^2 + y^2) \rho(x, y)\, dA$$

Moments of Inertia



$$I_x = \iint_D y^2 \rho(x, y)\, dA$$
$$I_y = \iint_D x^2 \rho(x, y)\, dA$$
$$I_0 = \iint_D (x^2 + y^2) \rho(x, y)\, dA$$

Find the moments of inertia I_x, I_y, and I_0 for the part of the disc of radius a in the first quadrant, assuming that the density is constant.