Math 213 - Triple Integrals in Cylindrical Coordinates

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Homework

- Re-read section 15.7
- Work on section 15.7, problems 1-13 (odd), 17-21 (odd) from Stewart
- Read section 15.8 for Monday November 5
- Begin webwork C3

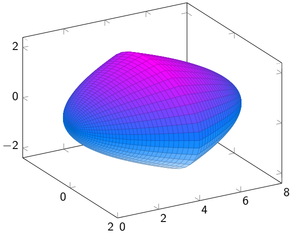
Unit III: Multiple Integrals

Lecture 24	Double Integrals over Rectangles
Lecture 25	Double Integrals over General Regions
Lecture 26	Double Integrals in Polar Coodinates
Lecture 27	Applications of Double Integrals
Lecture 28	Surface Area
Lecture 29	Triple Integrals
Lecture 30	Triple Integrals in Cylindrical Coordinates
Lecture 31	Triple Integrals in Spherical Coordinates
Lecture 32	Change of Variable in Multiple Integrals, Part I
Lecture 33	Change of Variable in Multiple Integrals, Part II
Lecture 34	Exam III Review

Goals of the Day

- Understanding Applications of Triple Integrals
- Know how to locate points and describe regions in cylindrical coordinates
- Know how to evaluate triple integrals in cylindrical coordinates

Volume



Recall that the area of a region D in the xy plane is given by the double integral

$$A(D) = \iint 1 \, dA$$

Similarly, the volume of a region E is the triple integral

$$V(E) = \iiint_E 1 \, dV$$

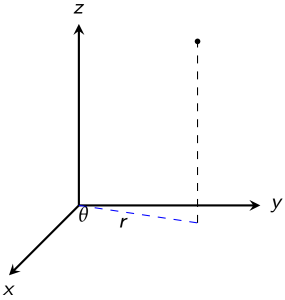
Find the volume of the solid enclosed by the paraboloids

$$y = x^2 + z^2$$

and

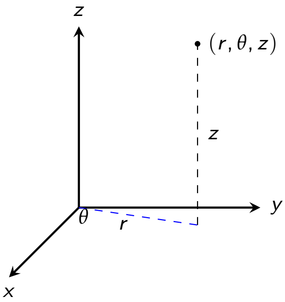
$$y = 8 - x^2 - z^2$$

Cylindrical Coordinates



Add the z-coordinate to polar coordinates and you get *cylindrical coordinates*

Cylindrical Coordinates



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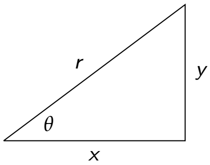

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$$r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x$$

$$x = r\cos\theta, \quad y = r\sin\theta$$

Cylindrical Coordinates

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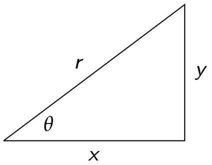
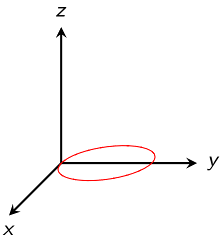


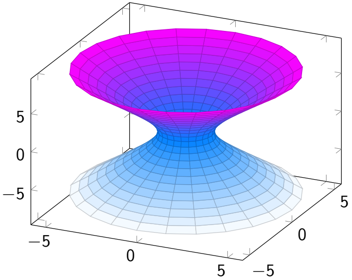
1. Find the cylindrical coordinates of the point $(-1, 1, 1)$
2. Find the cyindrical coordinates of the point $(-2, 2\sqrt{3}, 3)$
3. Find the rectangular coordinates of the point $(4, \pi/3, -2)$

Equations and Regions in Cylindrical Coordinates



1. Identify the surface $r = 2\sin\theta$

Equations and Regions in Cylindrical Coordinates



1. Identify the surface $r = 2\sin\theta$
2. Write the equation

$$2x^2 + 2y^2 - z^2 = 4$$

in cylindrical coordinates

Equations and Regions in Cylindrical Coordinates

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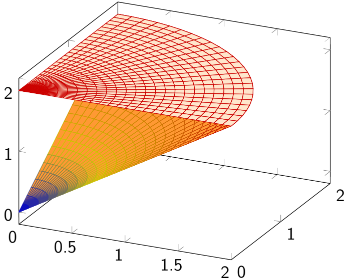
3. Sketch the solid described by the inequalities $0 \leq \theta \leq \pi/2$, $r \leq z \leq 2$

Equations and Regions in Cylindrical Coordinates



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Triple Integrals in Cylindrical Coordinates

In polar coordinates

$$dA = r\,dr\,d\theta$$

So, in cylindrical coordinates,

$$dV = r\,dr\,d\theta\,dz = r\,dz\,dr\,d\theta$$

If E is the region

$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

then

$$\iiint_E f(x,y,z)\,dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

If we can describe D in polar coordinates:

$$D = \{(r,\theta) : \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then we can evaluate

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

Step by Step

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

This formula summarizes a multi-step process. If

$$E = \{(x,y,z) : (x,y) \in D,\ u_1(x,y) \leq z \leq u_2(x,y)\}$$

then, to use the formula:

1. Substitute $x = r\cos\theta$, $y = r\sin\theta$ into u_1 and u_2 to find the limits of the inmost integral
2. Substitute $x = r\cos\theta$, $y = r\sin\theta$ into the formula for $f(x,y,z)$ to rewrite f as a function of r, θ, and z
3. After making these substitutions, evaluate the triple iterated integral

Triple Integrals in Cylindrical Coordinates

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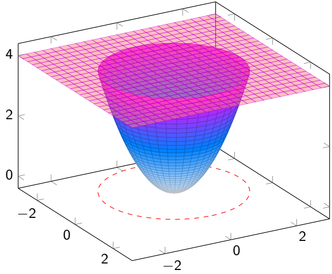

1. Find $\iiint_E z\,dV$ where E is enclosed by the paraboloid

$$z = x^2 + y^2$$

and the plane $z = 4$

Triple Integrals in Cylindrical Coordinates

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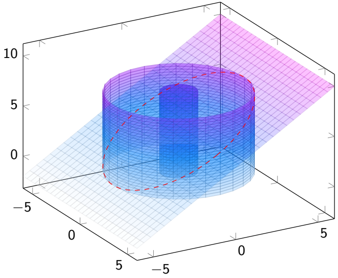

1. Find $\iiint_E z\,dV$ where E is enclosed by the paraboloid

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2. Find $\iiint_E (x-y)\,dV$ if E is the solid which lies between the cylinders

$$x^2 + y^2 = 1, \quad x^2 + y^2 = 16,$$

above the xy plane, and below the plane $z = y + 4$.