Math 213 - Vector Fields, Line Integrals

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Homework

- Prepare to Ace Exam III
- Finish Webwork C4
- Read Sections 16.1 and 16.2 for Friday
- Work on Stewart problems for 16.1 and 16.2:

  16.1: 11-18, 21, 23, 25, 29-32, 33
  16.2: 1-21 (odd), 33-41 (odd), 49, 50

Unit IV: Vector Calculus

Lecture 35	Vector Fields
Lecture 36	Line Integrals
Lecture 37	Line Integrals
Lecture 38	Fundamental Theorem
Lecture 39	Green's Theorem
Lecture 40	Curl and Divergence

Goals of the Day

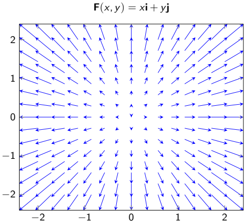
- Understand and Visualize Vector Fields
- Know what the *gradient vector field* of a function is
- Preview *line integrals*

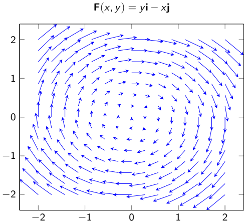
Vector Fields in the Plane

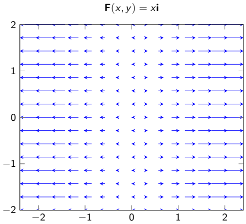
A **vector** field is a function that associates to each (x, y) a *vector*

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

We can visualize a vector field by a *field plot*





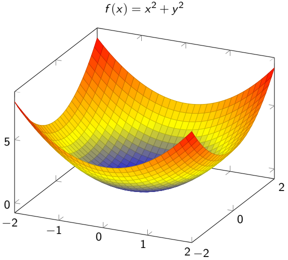


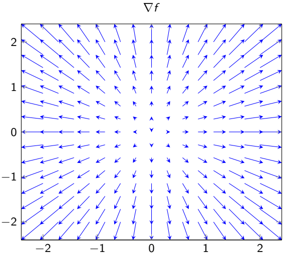
The Gradient Vector Field

If $f(x, y)$ is a function two variables, the *gradient vector field*

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

moves in the direction of greatest change of f





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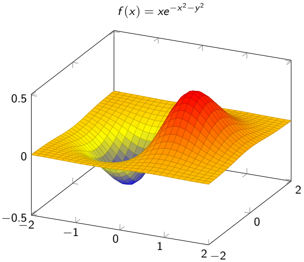
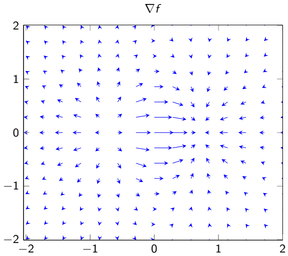
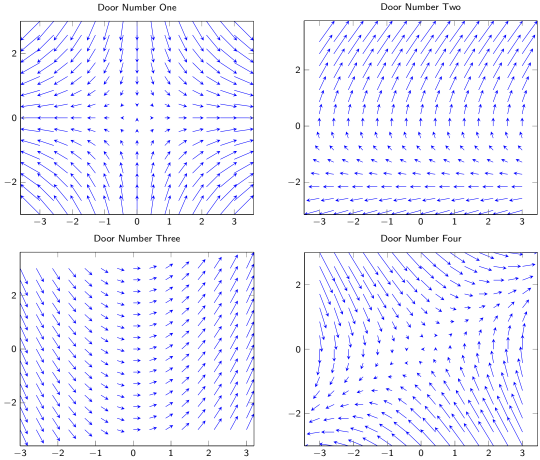




Mix and Match

Can you match the vector field with its field plot?

A	$\mathbf{F}(x,y) = \langle x, -y \rangle$	**B**	$\mathbf{F}(x,y) = \langle y, x-y \rangle$
C	$\mathbf{F}(x,y) = \langle y, y+2 \rangle$	**D**	$\mathbf{F}(x,y) = \langle \cos(x+y), x \rangle$

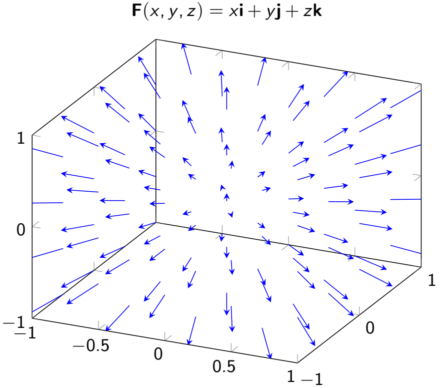


Vector Fields in Space

A **vector field** in space is a function that associates to each (x, y, z) a *vector*

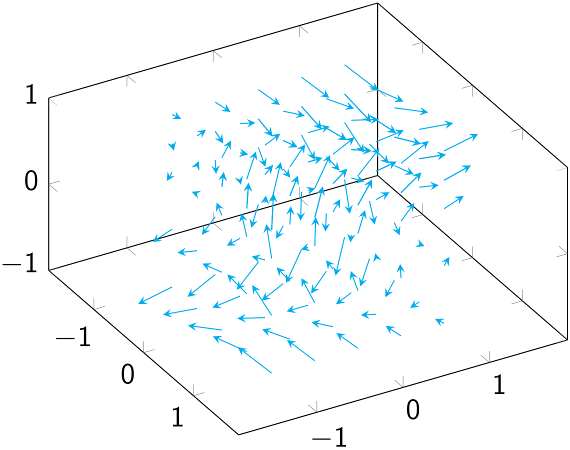
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

We can visualize a vector field by a *field plot*



Vector Fields in Space

$$\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$$



Vector Fields in Physics

1. The electric field generated by a point charge q at the origin is

$$\mathbf{E}(\mathbf{x}) = \frac{q\mathbf{x}}{|\mathbf{x}|^3}$$

2. The gravitational force exerted on a mass m at position $\mathbf{x}$ by a mass M at the origin is

$$\mathbf{F}(\mathbf{x}) = -\frac{GMm\mathbf{x}}{|\mathbf{x}|^3}$$

3. A *conservative force* $\mathbf{F}$ is the gradient of a *potential function* φ, i.e.,

$$\mathbf{F} = \nabla\phi$$

Preview: Line Integrals

Our next topic will be integrals of *scalar functions* and *vector functions* over curves in the plane and in space. If C is a curve in the plane or in space, we'll learn how to compute:

- $\int_C f(x,y)\,ds$, the integral of a scalar function over a plane curve C
- $\int_C \mathbf{F} \cdot d\mathbf{r}$, the integal of a vector function $\mathbf{F}(x,y)$ over a plane curve C
- $\int_C f(x,y,z)\,ds$, the integral of a scalar function over a space curve C
- $\int_C \mathbf{F} \cdot d\mathbf{r}$, the integral of a vector function $\mathbf{F}(x,y,z)$ over a space curve C

In all cases, we'll reduce these to Calculus I and II type integrals by parameterizing the curve C. We'll also learn how to compute integrals like

- $\int_C f(x,y)\,dx$
- $\int_C f(x,y)\,dy$

The Integral of a Scalar Function over a Plane Curve

If C is a plane curve, the **line integral of f along C** is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i$$

where we approximate the curve by n line segments of length Δs_i

As a practical matter, if C is parameterized by $(x(t), y(t))$ for $a \leq t \leq b$,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

so

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

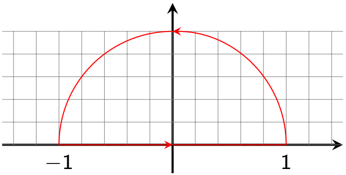
The Integral of a Scalar Function over a Plane Curve

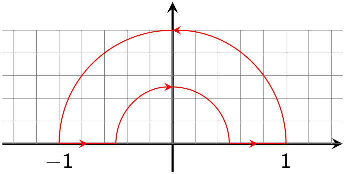
if C is parameterized by $(x(t), y(t))$ for $a \leq t \leq b$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. Find $\int_C (x/y)\, ds$ if C is the curve $x = t^2$, $y = 2t$ for $0 \leq t \leq 3$
2. Find $\int_C xy^4\, ds$ if C is the right half of the circle $x^2 + y^2 = 16$

Line Integrals over Piecewise Smooth Curves





A curve C is *piecewise smooth* if it is a union of smooth curves $C_1, \ldots C_n$. Some examples are shown at left.

If C consists of seveal smooth components, then

$$\int_C f(x,y)\,ds = \sum_{i=1}^{n} \int_{C_i} f(x,y)\,ds$$

Notice that each of these curves has an *orientation* that determines how the curve is parameterized–the parameterization should "follow the arrows."

1. Find $\int_C xy\,ds$ if C is the first curve shown at left.

Another Kind of Line Integral

For later use, we'll also need the line integral of f with respect to x and the line integral of f with respect to y:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt$$

1. Find $\int_C e^x\,dx$ if C is the arc of the curve $x = y^3$ from $(-1,-1)$ to $(1,1)$
2. Find $\int_C x^2\,dx + y^2\,dy$ if C is the arc of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$

Summary of Line Integrals in the Plane

If C is a parameterized curve $(x(t), y(t))$ where $a \leq t \leq b$:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt$$

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t)^2 + y'(t)^2}\,dt$$

Line Integrals in Space

If C is a space curve $(x(t), y(t), z(t))$ where $a \leq t \leq b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

1. Find $\int_C (x^2 + y^2 + z^2)\, ds$ if C is the space curve $(x(t), y(t), z(t)) = (t, \cos 2t, \sin 2t)$ for $0 leqt \leq 2\pi$

More Line Integrals in Space

Can you guess how to define $\int_C f(x, y, z)\, dx$, $\int_C f(x, y, z)\, dy$, and $\int_C f(x, y, z)\, dz$?

1. Find $\int_C (x + z)\, dx + \int_C (x + z)\, dy + \int_C (x + y)\, dz$ if C consists of the line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$