Math 213 - Linear Approximation

Peter A. Perry

University of Kentucky

September 27, 2019

Reminders

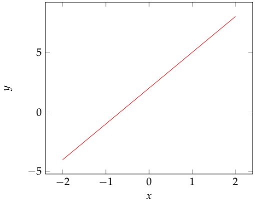
- Homework B2 on 14.3 is due tonight!

Unit II: Functions of Several Variables

Learning Goals

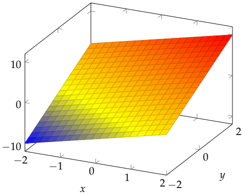
- Understand how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define the *tangent plane* to the graph of $z = f(x, y)$ at $(a, b, f(a, b))$
- Understand how the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ define the *linear approximation* $L(x, y)$ to $f(x, y)$ near $(x, y) = (a, b)$
- Understand the *total differential* dz of a function $z = f(x, y)$ and how it's used to compute percentage change and analyze error
- Generalize these ideas to functions of three variables

Warm-Up: Linear Functions



The graph of a line $Ax + By = C$ defines a *linear function* of one variable

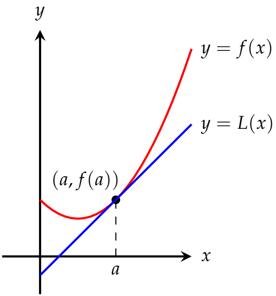
$$y = f(x) = \frac{C}{B} - \frac{A}{C}x$$



The graph of a plane $ax + by + cz = d$ defines a *linear function* of two variables

$$z = f(x, y) = \frac{d}{c} - \frac{a}{c}x - \frac{b}{c}y$$

Functions of One Variable - Tangent Line



The derivative $f'(a)$ gives the slope of the tangent line to the graph of $y = f(x)$ at $(a, f(a))$.

The derivative $f'(a)$ defines a linear function

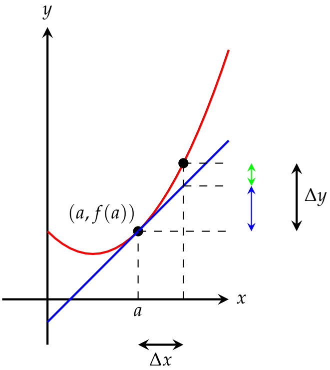
$$L(x) = f(a) + f'(a)(x - a)$$

the linear approximation to f near a

The differential of $y = f(x)$ is

$$dy = f'(x)\,dx$$

Functions of One Variable - Differentiability



Recall that if $y = f(x)$, the *increment* of y as x changes from a to $a + \Delta x$ is

$$\Delta y = f(a + \Delta x) - f(a).$$

If f is differentiable at a, then

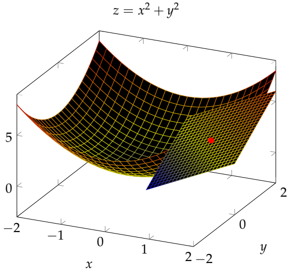
$$\Delta y = f'(a)\,\Delta x + \varepsilon \Delta x$$

where

$$\varepsilon \to 0 \text{ as } \Delta x \to 0$$

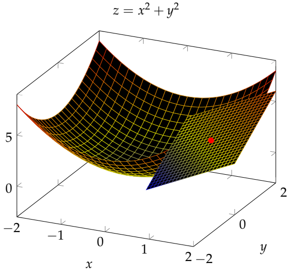
That is, the linear approximation is *very* good as $\Delta x \to 0$.

Derivatives - Two Variables



The derivatives $f_x(a, b)$ and $f_y(a, b)$ define a *tangent plane* to the graph of f at $(a, b, f(a, b))$

Derivatives - Two Variables



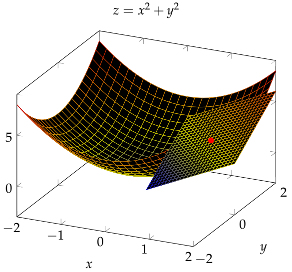
The derivatives $f_x(a,b)$ and $f_y(a,b)$ define a *tangent plane* to the graph of f at $(a,b,f(a,b))$

These derivatives define a linear function

$$\begin{aligned} L(x,y) = f(a,b) \\ + f_x(a,b)(x-a) \\ + f_y(a,b)(x-b) \end{aligned}$$

the linear approximation to f near (a,b)

Derivatives - Two Variables



The derivatives $f_x(a,b)$ and $f_y(a,b)$ define a *tangent plane* to the graph of f at $(a,b,f(a,b))$

These derivatives define a linear function

$$\begin{aligned} L(x,y) = f(a,b) \\ &+ f_x(a,b)(x-a) \\ &+ f_y(a,b)(x-b) \end{aligned}$$

the linear approximation to f near (a,b)

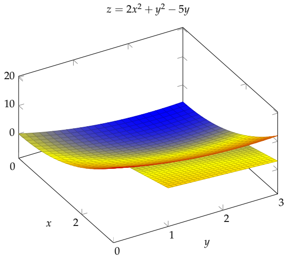
The differential of $z = f(x,y)$ is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Find the Tangent Plane

If f has continuous partial derivatives, the tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$



1. Find the equation of the tangent plane to the surface

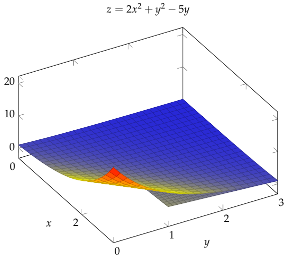
$$z = 2x^2 + y^2 - 5y$$

at $(1, 2, -4)$.

Find the Tangent Plane

If f has continuous partial derivatives, the tangent plane to $z = f(x, y)$ at $(a, b, f(a, b))$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$



1. Find the equation of the tangent plane to the surface

$$z = 2x^2 + y^2 - 5y$$

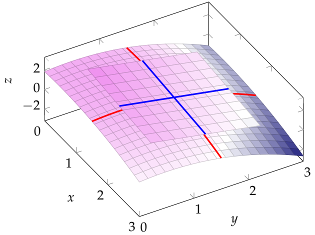
at $(1, 2, -4)$.

2. Find the equation of the tangent plane to the surface

$$z = e^{x-y}$$

at $(2, 2, 1)$.

The Tangent Plane Contains Tangent Lines



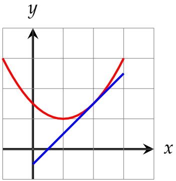
The red curves represent $f(a,y)$ and $f(x,b)$

The blue lines are the tangent lines

$$r_1(t) = \langle a, b, f(a,b) \rangle + t\langle 1, 0, f_x(a,b) \rangle$$

$$r_2(t) = \langle a, b, f(a,b) \rangle + t\langle 0, 1, f_y(a,b) \rangle$$

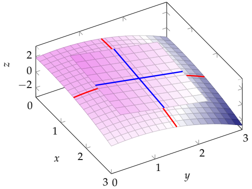
The Tangent Plane Defines a Linear Approximation



The tangent line is the graph of a linear function

$$L(x) = f(a) + f'(a)(x - a)$$

that approximates $f(x)$ near $x = a$



The tangent plane is the graph of a linear function

$$L(x,y) = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b)$$

that approximates $f(x,y)$ near $(x,y) = (a,b)$

The Linear Approximation

The linear approximation to $f(x,y)$ at (a,b) is

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

1. Show that the linear approximation to $f(x,y) = e^x \cos(xy)$ at $(0,0)$ is $L(x,y) = x+1$
2. Suppose that $f(2,5) = 6$, $f_x(2,5) = 1$, and $f_y(2,5) = -1$. Use a linear approximation to estimate $f(2.2, 4.9)$

Differentiability

If $z = f(x,y)$, the *increment* of z as x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$ is:

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a,b)$$

f is *differentiable* at (a,b) if

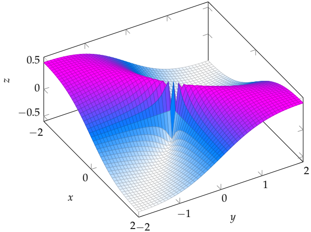
$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where ε_1 and ε_2 approach 0 as $(\Delta x, \Delta y) \to (0,0)$.

Theorem If the partial derivatives f_x and f_y of f exist near (a,b), and are continuous at (a,b), then f is differentiable at (a,b).

1. Explain why the function $f(x,y) = \sqrt{xy}$ is differentiable at $(1,4)$ and find its linearization

What Happens if f is not differentiable?



Let

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Use the definition to check that $f_x(0,0)=f_y(0,0)=0$

Show that $f_x(x,y)$ and $f_y(x,y)$ are not continuous at $(0,0)$

Differentials

For a function f of one variable, the differential of $y = f(x)$ is given by

$$dy = f'(x)\, dx$$

For a function f of two variables, the differential of $z = f(x, y)$ is

$$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

---

1. The radius of a circle is measured as 10cm with an error of at most 0.2cm. What is the maximum calculated area of the circle?

2. The length and width of a rectangle are measured as 30cm and 24cm, with an error of at most 0.1cm each. What is the maximum error in the calculated area of the rectangle?

Three Variables

If $w = f(x, y, z)$:

- The *linear approximation* of f at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - a) + f_z(a, b, c)(z - c)$$

- The *increment* of w is

$$\Delta w = f(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

- The *differential* dw is

$$dw = \frac{\partial w}{\partial x}\, dx + \frac{\partial w}{\partial y}\, dy + \frac{\partial w}{\partial z}\, dz$$

Three Variables

The *linear approximation* of f at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - a) + f_z(a, b, c)(z - c)$$

Find the linear approximation to

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

at $(x, y, z) = (3, 2, 6)$ and estimate

$$\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$$

Summary

- We showed how the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ determine the tangent plane to the graph of f at $(a,b,f(a,b))$
- We saw how the tangent plane defines a linear approximation $L(x,y)$ to $f(x,y)$ near $(x,y)=(a,b)$
- We saw how these ideas generalize to functions of three variables